Solution to Problems ♠–10

Problem A: *How many solutions in nonnegative integers are there to the equation*

$$x_1 + x_2 + x_3 + x_4 = 105 \quad ?$$

Answer:

Every solution to our equation (in nonnegative integers) corresponds to placing 3 dividers in a sequence of 105 stones, thus creating 4 bins.

$$x_1 \text{ stones } \Big| x_2 \text{ stones } \Big| x_4 \text{ stones } \Big| x_4 \text{ stones}$$

And vice versa. (Note: if $x_i = 0$ then two dividers $\Big|$ are adjacent.) Thus the problem asks in how many ways we may place 3 dividers between 105 objects. This amounts to choosing 3 spots out of $105 + 3$ places, so it can be done in

$$\binom{105+3}{3} = \frac{108!}{3! \cdot 105!} = \frac{108 \cdot 107 \cdot 106}{6} = 204156$$

ways.

Correct solutions were received from :

(1) Brad Tuttle

Problem B: *Find the absolute maximum value of the function*

$$f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-2|}, \qquad x \in \mathbb{R}.$$

Answer: We note that

$$f(x) = \begin{cases} \frac{1}{1-x} + \frac{1}{1-(x-2)} & \text{if } x < 0, \\ \frac{1}{1+x} + \frac{1}{1-(x-2)} & \text{if } 0 \le x < 2, \\ \frac{1}{1+x} + \frac{1}{1+(x-2)} & \text{if } x \ge 2, \end{cases}$$

and clearly the function f is continuous on its domain $\mathbb{R}$ and differentiable on the intervals $(-\infty, 0)$, $(0, 2)$ and $(2, \infty)$. Moreover,

$$f'(x) = \begin{cases} \frac{1}{(1-x)^2} + \frac{1}{(3-x)^2} & \text{if } x < 0, \\ \frac{-1}{(1+x)^2} + \frac{1}{(3-x)^2} & \text{if } 0 < x < 2, \\ \frac{-1}{(1+x)^2} - \frac{1}{(x-1)^2} & \text{if } x > 2. \end{cases}$$

Plainly,

if $x < 0$ then $f'(x) > 0$,

if $x > 2$ then $f'(x) < 0$.

Also, for $x \in (0, 2)$ we have

$$f'(x) = \frac{-1}{(1+x)^2} + \frac{1}{(3-x)^2} = \frac{8(x-1)}{(3-x)^2 \cdot (x+1)^2}.$$

Therefore,

if $0 < x < 1$ then $f'(x) < 0$,

if $1 < x < 2$ then $f'(x) > 0$, and

$f'(1) = 0$.

Consequently, by the First Derivative Test, the local maxima of the function f are at $x = 0$ and $x = 2$, where takes the value $4/3$. Therefore, $4/3$ is the absolute maximum value of f.

Correct solutions were received from :

(1) Brad Tuttle — POW 10B: ♠