Solution to Problems ♠–1

Problem A: *A graph has 2018 vertices. Given any four vertices, there is at least one joined to the other three. What is the smallest number of vertices which are joined to all other 2017 vertices?*

Answer: 2015

Suppose towards contradiction that there were 4 vertices not connected to all other 2017 vertices. Let one of them be A. Take a vertex B not joined to A.

Now if X and Y are any two other vertices, X and Y must be connected, because otherwise none of the points A, B, X, Y could be joined to the other 3. By our assumption, there must be two other vertices C and D not joined to 2017 vertices. We have just shown that C must be joined to every vertex except possibly A and B. So it must be not joined to one of those. Similarly for D. But now none of A, B, C, D is joined to the other 3, a contradiction.

So there cannot be 4 vertices not connected to 2017 vertices. But there can be 3. Just take the graph to have all edges except AB and AC.

Correct solution was received from :

(1) Matthew Arnold — POW 1A: ♠
(2) Brad Tuttle — POW 1A: ♠

Problem B: *A graph has $n > 2$ vertices. Show that we can find two vertices A and B such that at least $\lfloor n/2 \rfloor - 1$ of the remaining vertices are connected to either both or neither of A and B.*

Answer: Let $\mathcal{Z}$ be the collection of all pairs $(X, \{Y, Z\})$, where X, Y, Z are distinct vertices such that X is connected to just one of Y, Z.

If a vertex X_0 is connected to exactly k vertices, then there are exactly $k(n-1-k)$ pairs $(X_0, \{Y, Z\}) \in \mathcal{Z}$. Since $k(n-1-k) \leq \frac{(n-1)^2}{4}$ (remember $4ab \leq (a+b)^2$), we conclude that the set $\mathcal{Z}$ has at most $\frac{n(n-1)^2}{4}$ elements.

There are $n(n-1)/2$ possible two element sets of vertices $\{Y, Z\}$. Therefore there is a set $\{A, B\}$ which appears as the second coordinate of at most $\lfloor (n-1)/2 \rfloor$ elements of $\mathcal{Z}$. Then there are at least $n - 2 - \lfloor (n-1)/2 \rfloor = \lfloor n/2 \rfloor - 1$ vertices X which are connected to both of A and B or to neither of A and B.

No correct solutions received