Problem of the week #6: Solutions

The divergent sums we want meaningful values for are

\[ A = 1 + 2 + 3 + 4 + \cdots \]
\[ B = 1 + \frac{2^2}{2} + \frac{3^2}{3} + \frac{4^2}{4} + \cdots \]
\[ C = 1 + \frac{2^3}{3} + \frac{3^3}{3} + \frac{4^3}{4} + \cdots \]

**Solution 1.** Differentiating the geometric series

\[ \frac{x}{1+x} = x - x^2 + x^3 - x^4 + \cdots \]

and then multiplying by \( x \) gives

\[ \frac{x}{(1+x)^2} = x - 2x^2 + 3x^3 - 4x^4 + \cdots \]

Doing this twice more yields two more series expansions

\[ \frac{x(1-x)}{(1+x)^3} = x - 2^2x^2 + 3^2x^3 - 4^2x^4 + \cdots \]
\[ \frac{1-4x+x^2}{(1+x)^4} = x - 2^3x^2 + 3^3x^3 - 4^3x^4 + \cdots \]

Setting \( x = 1 \) in the last three equations yields

\[ \frac{1}{4} = 1 - 2 + 3 - 4 + \cdots \]
\[ 0 = 1 - 2^2 + 3^2 - 4^2 + \cdots \]
\[ -\frac{1}{8} = 1 - 2^3 + 3^3 - 4^3 + \cdots \]

To obtain the regularized values of the non-alternating versions of these sums, we may use zero-padding and linearity. In particular,

\[ a_1 + a_2 + a_3 + a_4 + \cdots = 0 + a_1 + 0 + a_2 + 0 + a_3 + 0 + \cdots \]
Then we may rewrite
\[
\frac{1}{4} = 1 - 2 + 3 - 4 + \ldots
\]
\[
= (1 + 2 + 3 + 4 + \ldots) - 2(0 + 2 + 0 + 4 + \ldots)
\]
\[
= (1 + 2 + 3 + 4 + \ldots) - 4(0 + 1 + 0 + 2 + \ldots)
\]
\[
= A - 4A = -3A
\]
which implies \( A = -\frac{1}{12} \) and then
\[
0 = 1 - 2^2 + 3^2 - 4^2 + \ldots
\]
\[
= (1 + 2^2 + 3^2 + 4^2 + \ldots) - 2(0 + 2^2 + 0 + 4^2 + \ldots)
\]
\[
= (1 + 2^2 + 3^2 + 4^2 + \ldots) - 8(0 + 1^2 + 0 + 2^2 + \ldots)
\]
\[
= B - 8B = -7B
\]
which implies \( B = 0 \) and then
\[
-\frac{1}{8} = 1 - 2^3 + 3^3 - 4^3 + \ldots
\]
\[
= (1 + 2^3 + 3^3 + 4^3 + \ldots) - 2(0 + 2^3 + 0 + 4^3 + \ldots)
\]
\[
= (1 + 2^3 + 3^3 + 4^3 + \ldots) - 16(0 + 1^3 + 0 + 2^3 + \ldots)
\]
\[
= C - 16C = -15C
\]
which implies \( C = \frac{1}{120} \).
Solution 2. We may find $A, B, C$ and their alternating versions, which we’ll call $X, Y, Z$, without differentiating the geometric series formula.

\[
W = 1 - 1 + 1 - 1 + \cdots
\]

\[
X = 1 - 2 + 3 - 4 + \cdots
\]

\[
Y = 1 - 2^2 + 3^2 - 4^2 + \cdots
\]

\[
Z = 1 - 2^3 + 3^3 - 4^3 + \cdots
\]

The geometric series formula already gives $W = \frac{1}{2}$, but also

\[
2W = \frac{(1 - 1 + 1 - 1 + \cdots) + (0 + 1 - 1 + 1 - \cdots)}{2} = 1
\]

which implies $W = \frac{1}{2}$ as well. Similarly,

\[
2X = \frac{(1 - 2 + 3 - 4 + \cdots) + (0 + 1 - 2 + 3 - \cdots)}{2} = W
\]

and $2X = \frac{1}{2}$ implies $X = \frac{1}{4}$.

The next one requires splitting up into two previous alternating sums:

\[
2Y = \frac{(1 - 2^2 + 3^2 - 4^2 + \cdots) + (0 + 1^2 - 2^2 + 3^2 - \cdots)}{2} = 1 - 3 + 5 - 7 + \cdots
\]

\[
= \frac{(1 - 1 + 1 - 1 + \cdots) - 2(0 + 1 - 2 + 3 - \cdots)}{2} = W - 2X
\]

and $2Y = \frac{1}{2} - 2(\frac{1}{4})$ implies $Y = 0$.

And the last one requires splitting into three previous sums.
\begin{align*}
2Z &= (1 - 2^3 + 3^3 - 4^3 + 5^3 - \ldots) \\
    &\quad + (0 + 1^3 - 2^3 + 3^3 - 4^3 + \ldots) \\
    &= 1 - 7 + 19 - 37 + 61 - \ldots \\
    &= (1 - 1 + 1 - 1 + \ldots) \\
    &\quad - 6(0 + 1 - 3 + 6 - 10 + \ldots) \\
\end{align*}

Notice 1, 3 = 1 + 2, 6 = 1 + 2 + 3, 10 = 1 + 2 + 3 + 4 are the triangular numbers, which satisfy \(1 + 2 + 3 + \cdots + n = \frac{1}{2}(n^2 + n)\). Continuing,

\begin{align*}
(1 - 1 + 1 - 1 + \ldots) \\
    &= -3(0 + 1 - 2 + 3 - 4 + \cdots) \\
    &\quad - 3(0 + 1 - 4 + 9 - 16 + \cdots) \\
    &= W - 3X - 3Y \\
\end{align*}

and \(2Z = \frac{1}{2} - 3(\frac{1}{4}) - 3(0)\) implies \(Z = -\frac{1}{8}\).

We could have also said \((n + 1)^3 - n^3 = 3n^2 + 3n + 1\) to similar effect.

Then \(A, B, C\) can be gotten from \(X, Y, Z\) as in Solution 1.

The Riemann zeta function, defined by \(\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\) for \(\text{Re}(s) > 1\) (its abscissa of convergence), also exists for complex numbers \(s\) with real parts less than or equal to 1, except at \(s = 1\) itself. This is like the situation for \((1 - x)^{-1} = \sum_{n=1}^{\infty} x^n\), where the series converges for \(|x| < 1\) but the function exists for all \(x\) except \(x = 1\). The process of extending the domain of a function in the complex plane is called **analytic continuation**, common in regularization.

In general, \(\zeta(-n) = 1 + 2^n + 3^n + 4^n\) have the formula \(\zeta(-n) = B_{n+1}/(n + 1)\), where the **Bernoulli numbers** \(B_n\) appear in the exponential generating function \(x/(e^x - 1) = \sum_{n=0}^{\infty} (B_n/n!)x^n\), as well as coefficients in all of the so-called Faulhaber polynomials \(P_s(n)\) defined by \(P_s(n) = \sum_{k=1}^{n} k^s = 1 + 2^s + 3^s + \cdots + n^s\), e.g. \(P_1(n) = \frac{1}{2}(n^2 + n)\).