Problem of the week #2: Background

A polynomial in multiple variables is called homogeneous if every monomial in it has the same total degree. So $x^2 + xy + y^2$ is homogeneous while $x^2 + y$ is not, for example. There is a one-to-one correspondence between polynomials in one variable and homogeneous polynomials in two variables of a given degree. For example,

\[
x^2 + 1 \leftrightarrow x^2 + y^2 \\
x^2 + 2x + 1 \leftrightarrow x^2 + 2xy + y^2 \\
4x^3 - 3x \leftrightarrow 4x^3 - 3xy^2
\]

To turn a degree-$n$ polynomial $f(x)$ into $g(x, y)$, multiply $f(x/y)$ by $y^n$ so all negative powers of $y$ are cancelled. For example

\[
y^3 \left[ 4 \left( \frac{x}{y} \right)^3 - 3 \left( \frac{x}{y} \right) \right] = 4x^3 - 3xy^2.
\]

Conversely, to turn $g(x, y)$ into $f(x)$ simply evaluate at $y = 1$. These processes are called homogenization and dehomogenization. They may also be applied to rational functions as well.

The geometric sum formula is given by

\[
S = 1 + x + \cdots + x^n = \frac{x^n - 1}{x - 1},
\]

which follows by comparing $S$ with $xS$. Homogenizing yields

\[
x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n = \frac{x^{n+1} - y^{n+1}}{x - y}.
\]