Problem of the week #1: Solutions

\[ S(t) = \int_{0}^{t} \frac{d\tau}{\cosh \tau}, \quad T(s) = \int_{0}^{s} \frac{d\sigma}{\cos \sigma}. \]

The function \( S(t) \) is called the **Gudermannian function** and \( T(s) \) the inverse Gudermannian function. Writing \( S = S(T) \) or \( T = T(S) \), they satisfy the following three equivalent identities:

\begin{align*}
1 & \quad \tan S = \sinh T \\
2 & \quad \sin S = \tanh T \\
3 & \quad \cos S = \sech T
\end{align*}

For example, (1) says \( \tan S(t) = \sinh t \) and \( \tan s = \sinh T(s) \).

One identity may be converted into another by applying transformations, using circular Pythagorean identities on the left and hyperbolic Pythagorean identities on the right. Applying \( \sqrt{1 - x^2} \) converts between (2) and (3); applying \( x/\sqrt{1 + x^2} \) converts from (1) to (2) and its inverse \( x/\sqrt{1 - x^2} \) from (2) to (1); applying \( 1/\sqrt{1 + x^2} \) converts from (1) to (3) and its inverse \( \sqrt{1 - x^2}/x \) from (3) to (1). When converting from (3) it suffices to assume \( S, T \geq 0 \) since they are odd functions.

There is also a fourth equivalent half-angle identity

\[ (4) \quad \tan \left( \frac{S}{2} \right) = \tanh \left( \frac{T}{2} \right). \]

This follows from all (hence any) of (1),(2),(3) using either version of \( \tan \) and \( \tanh \)'s half-angle formulas. For example,

\[ \tan \frac{S}{2} = \frac{\sin S}{1 + \cos S} = \frac{\tanh T}{1 + \sech T} = \frac{\sinh T}{\cosh T + 1} = \tanh \frac{T}{2}. \]

Conversely, (4) may be converted to (1) by applying \( 2x/(1 - x^2) \), to (2) by applying \( 2x/(1 + x^2) \), and to (3) by applying \( (1 - x^2)/(1 + x^2) \); therefore all of the identities (1),(2),(3),(4) are equivalent to each other.

To show \( S(t) \) and \( T(s) \) are inverse functions, it suffices to establish any of the four identities for \( S(t) \) and \( t \) and any other one of the four for \( s \) and \( T(s) \). For example, if evaluating \( S(t) \) yields (1) and evaluating \( T(s) \) yields (2), then (1) implies (2) so \( S(t) = \sin^{-1}(\tanh t) \) and \( T(s) = \tanh^{-1}(\sin s) \) and hence they are inverse functions.
**Cofunction substitutions.** Evaluate the definite integral $S(t)$ using the substitution $u = \sinh(\tau)$ (where $du = \cosh(\tau)d\tau$) and the hyperbolic Pythagorean identity $\cosh^2 - \sinh^2 = 1$:

$$S(t) = \int_0^t \frac{\cosh(\tau)d\tau}{1 + \sinh^2(\tau)} = \int_0^{\sinh t} \frac{du}{1 + u^2} = \tan^{-1}(\sinh(t)).$$

Evaluate $T(s)$ first by using the substitution $u = \sin(\sigma)$ (where $du = \cos(\sigma)du$) and the circular Pythagorean identity $\cos^2 + \sin^2 = 1$:

$$T(s) = \int_0^s \frac{\cos(\sigma)d\sigma}{1 - \sin^2(\sigma)} = \int_0^{\sin s} \frac{du}{1 - u^2} = \tanh^{-1}(\sin(s)).$$

Without directly knowing or recognizing the derivative of $\tanh^{-1}$, it is also possible to use partial fraction decomposition:

$$\int_0^{\sin s} \frac{1}{2} \left( \frac{1}{1 - u} + \frac{1}{1 + u} \right) du = \frac{1}{2} \ln \left| \frac{1 + \sin s}{1 - \sin s} \right| = \tanh^{-1}(\sin(s)).$$

**Exponential substitutions.** Evaluate $S(t)$ using the substitution $u = e^\tau$ (where $du = e^\tau d\tau$) and absorbing the interval over which it is taken (since $\cosh$ is an even function):

$$S(t) = \int_0^t \frac{2d\tau}{e^\tau + e^{-\tau}} = \int_0^t \frac{e^\tau d\tau}{e^{2\tau} + 1} = \int_{e^{-t}/2}^{e^t/2} \frac{du}{u^2 + 1} = \tan^{-1}(e^t) - \tan^{-1}(e^{-t}).$$

Apply tangent with difference-angle identity to get

$$\tan S(t) = \frac{e^t - e^{-t}}{1 + e^t e^{-t}} = \sinh(t).$$

We may instead have chosen to divide $S$ by 2 in which case

$$\frac{S(t)}{2} = \int_0^t \frac{d\tau}{e^\tau + e^{-\tau}} = \int_{1}^{e^t/2} \frac{du}{u^2 + 1} = \tan^{-1}(e^t) - \tan^{-1}(1).$$

Applying tangent (and multiplying by $e^{-t/2}/e^{-t/2}$) yields

$$\tan \frac{S(t)}{2} = \frac{e^t - 1}{1 + e^t} = \frac{(e^{t/2} - e^{-t/2})/2}{(e^{t/2} + e^{-t/2})/2} = \tanh(t/2).$$
Phasor substitutions. With functions of complex variables and path integrals in the complex plane it is possible to evaluate $T(s)$ using the substitution $u = e^{i\sigma}$ (where $du = ie^{i\sigma}d\sigma$) alongside the formula $\cos(\sigma) = (e^{i\sigma} + e^{-i\sigma})/2$. Absorbing 2 into the integral yields:

$$T(s) = \int_{0}^{s} \frac{2d\sigma}{e^{i\sigma} + e^{-i\sigma}} = \int_{-s}^{s} \frac{e^{i\sigma} d\sigma}{e^{2i\sigma} + 1} = \frac{1}{i} \int_{e^{-is}}^{e^{is}} \frac{du}{u^2 + 1} = \frac{\tan^{-1}(e^{is}) - \tan^{-1}(e^{-is})}{i}.$$

Apply tangent with difference angle identity to get

$$\tan iT(s) = \frac{e^{is} - e^{-is}}{1 + e^{is}e^{-is}} = i\sin s,$$

which is equivalent to $\tanh T(s) = \sin s$. Dividing by 2 instead,

$$\frac{T(s)}{2} = \int_{0}^{s} \frac{d\sigma}{e^{i\sigma} + e^{-i\sigma}} = \frac{1}{i} \int_{0}^{e^{is}} \frac{du}{u^2 + 1} = \frac{\tan^{-1}(e^{is}) - \tan^{-1}(1)}{i}.$$

Applying tangent (and multiplying by $e^{-is/2}/e^{-is/2}$) yields

$$\frac{\tan iT(s)}{2} = \frac{e^{is} - 1}{1 + e^{is}} = \frac{e^{is/2} - e^{-is/2}}{e^{is/2} + e^{-is/2}} = \tan(s/2).$$

Differentiation. Another idea: we may show $(T \circ S)(t) = t$ by showing both sides have the same derivative and agree at the initial value $(T \circ S)(0) = 0$. Differentiating with the chain rule and solving for $S$ indicates we need to show $S(t) = \pm \cos^{-1}(\text{sech}(t))$. This, again, can be argued by showing both sides are equal at $t = 0$ and have the same derivative, though one needs to manage the continuity of the ± sign, and then the same can be done to show $(S \circ T)(s) = s$, or else argue $S$ and $T$ are one-to-one because they are monotonic because their integrands are always positive on $S$ and $T$’s domains.