Yoga of π: Solution

The first trick is that we can rewrite a positive constant α as $\alpha = \int_{0\le v\le\alpha} \mathrm{d}v$, or equivalently $1/\alpha = \int_{0\le\alpha v\le 1} \mathrm{d}v$, which we can use to rewrite the integrand as its own integral ($\alpha = u^2+1$, ignoring the 2), creating a double integral:

$$\int\limits_{-1\le u\le 1} \frac{2\,\mathrm{d}u}{u^2+1} \quad = \iint\limits_{\substack{-1\le u\le 1\\ 0\le(u^2+1)v\le 1}} 2\,\mathrm{d}u\mathrm{d}v$$

Observe the resemblance of $(u^2+1)v = u^2v+v$ to x^2+y^2; both are bounded by the same inequality. This suggests setting $u^2v = x^2$ and $v = y^2$, meaning

$$\begin{cases} x = u\sqrt{v} \\ y = \sqrt{v} \end{cases} \iff \begin{cases} u = x/y \\ v = y^2 \end{cases}$$

With this change of variables, the double integral becomes

$$\iint\limits_{\substack{-1\le u\le 1\\ 0\le(u^2+1)v\le 1}} 2\,\mathrm{d}u\mathrm{d}v \;= \iint\limits_{\substack{-1\le x/y\le 1\\ x^2+y^2\le 1}} 2\,\frac{\partial(u,v)}{\partial(x,y)}\mathrm{d}x\mathrm{d}y$$

We may calculate the Jacobian determinant to be

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1/y & -x/y^2 \\ 0 & 2y \end{vmatrix} = 2$$

and therefore the double integral becomes

$$\iint\limits_{\substack{-y\le x\le y\\ x^2+y^2\le 1}} 4\,\mathrm{d}x\mathrm{d}y = \iint\limits_{Q_2} \mathrm{d}x\mathrm{d}y + \iint\limits_{Q_2} \mathrm{d}x\mathrm{d}y + \iint\limits_{Q_2} \mathrm{d}x\mathrm{d}y + \iint\limits_{Q_2} \mathrm{d}x\mathrm{d}y.$$

The domain Q_2 is the top quarter sector of the unit disk. We can use the substitution $(x,y)\mapsto(y,x)$ to turn two of the Q_2 domains into Q_1, the right quarter sector. Then, we can use the substitution $(x,y)\mapsto(-x,-y)$ to turn one Q_1 and one Q_2 into the other two quarters Q_3 and Q_4. Also note these substitutions do not change the integrands or differentials.

Thus the four integrals become

$$\iint\limits_{Q_1} \mathrm{d}x\mathrm{d}y + \iint\limits_{Q_2} \mathrm{d}x\mathrm{d}y + \iint\limits_{Q_3} \mathrm{d}x\mathrm{d}y + \iint\limits_{Q_4} \mathrm{d}x\mathrm{d}y = \iint\limits_{x^2+y^2\leq 1} \mathrm{d}x\mathrm{d}y.$$

And so we have followed the rules to conclude

$$\int\limits_{-1\leq u\leq 1} \frac{2\,\mathrm{d}u}{u^2+1} \quad\longrightarrow\quad \iint\limits_{x^2+y^2\leq 1} \mathrm{d}x\mathrm{d}y.$$
