Lazy Spline: Solution

Denote the central vertex $\mathbf{v}$. The curve is given by the formula

$$\mathbf{x}(t) = t^4\,\mathbf{a} + 4t^3(1-t)\,\mathbf{b} + 6t^2(1-t)^2\,\mathbf{v} + 4t(1-t)^3\,\mathbf{c} + (1-t)^4\,\mathbf{d}.$$

with velocity vector $\mathbf{x}'(t) = \mathbf{p}(t) + q(t)\mathbf{v}$, where

$$\mathbf{p}(t) = 4t^3\,\mathbf{a} + 4(3-4t)t^2\,\mathbf{b} + 4(1-4t)(1-t)^2\,\mathbf{c} - 4(1-t)^3\,\mathbf{d}$$

and $q(t) = 12t(2t^2 - 3t + 1)$, computed by W|A (because we're lazy, but certainly possible to compute by hand). Then the energy functional is

$$E = \int_0^1 \|\mathbf{p}(t) + q(t)\mathbf{v}\|^2\,\mathrm{d}t$$

which, using $\|\mathbf{r}\|^2 = \mathbf{r}\cdot\mathbf{r}$ and FOILing out becomes

$$E = \int_0^1 \|\mathbf{p}(t)\|^2 + 2\mathbf{p}(t)\cdot q(t)\mathbf{v} + q(t)^2\|\mathbf{v}\|^2\ \mathrm{d}t$$

This is extremized when $\nabla E = \mathbf{0}$, where E is a scalar function of $\mathbf{v}$. And

$$\nabla E = \int_0^1 2\mathbf{p}(t)q(t) + 2q(t)^2\mathbf{v}\ \mathrm{d}t.$$

Setting $\nabla E = \mathbf{0}$ and solving for $\mathbf{v}$ yields

$$\mathbf{v} = -\Big(\int_0^1 \mathbf{p}(t)q(t)\,\mathrm{d}t\Big)\Big/\Big(\int_0^1 q(t)^2\,\mathrm{d}t\Big)$$

Again using software we compute all integrals and coefficients and get

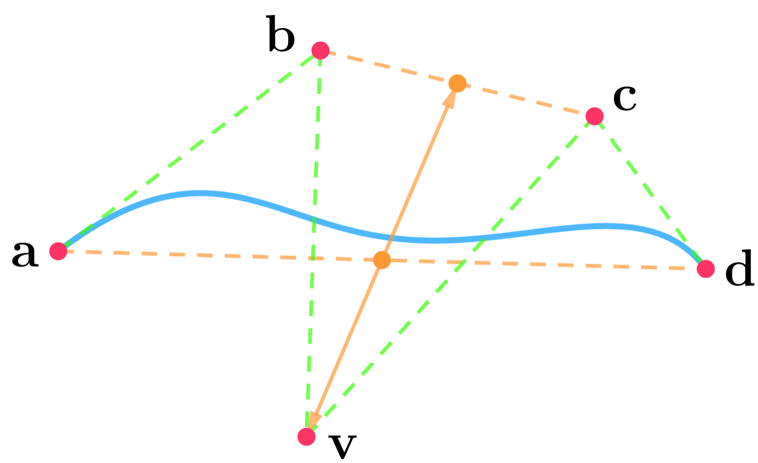
$$-\frac{(-24/35)\,\mathbf{a} + (12/35)\,\mathbf{b} + (12/35)\,\mathbf{c} - (24/35)\,\mathbf{d}}{24/35}$$

which simplifies to $\mathbf{v} = \mathbf{a} - \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{c} + \mathbf{d}$.

The solution $\mathbf{v}$ has the following interpretation: it is the midpoint of BC reflected across the midpoint of AD. This is evident in the formula

$$\mathbf{v} = \tfrac{1}{2}(\mathbf{a} + \mathbf{d}) - \left(\tfrac{1}{2}(\mathbf{b} + \mathbf{c}) - \tfrac{1}{2}(\mathbf{a} + \mathbf{d})\right)$$

Essentially, while $\mathbf{b}$ and $\mathbf{c}$ draw the curve outwards, the solution $\mathbf{v}$ is located so as to pull the curve equally and oppositely back inward.



Quadratic and cubic Bèzier curves are used in **vector graphics**.

While common filetypes like PNG and JPG store a rectangular array of pixels, other filetypes like SVG store equations that describe curves and gradients; the latter kind are *scalable* - they look smooth even zoomed-in.