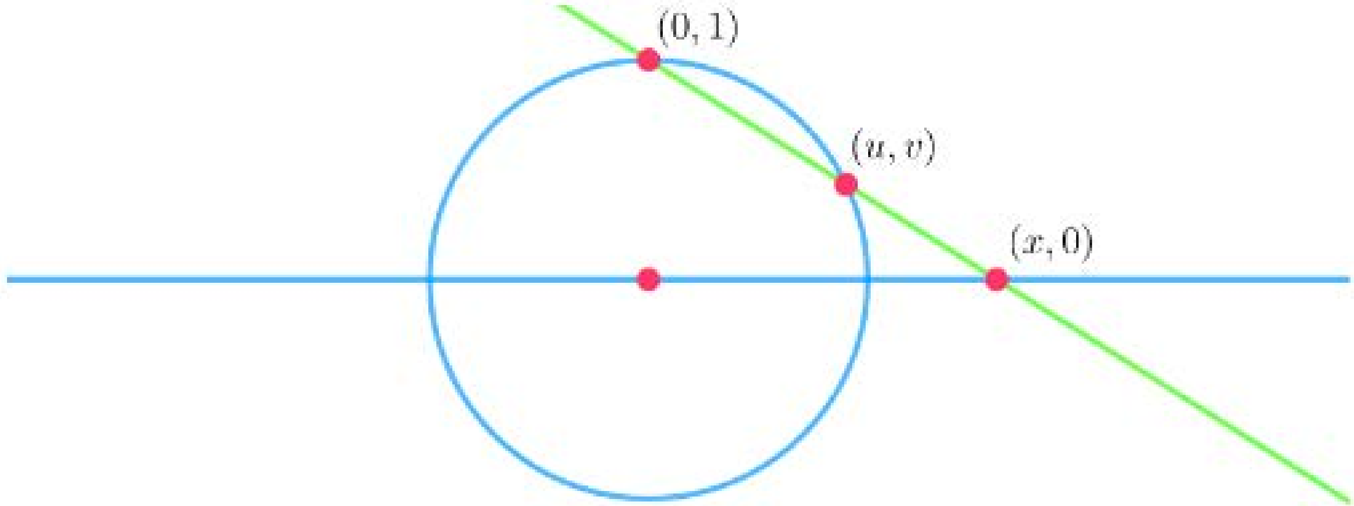
Quadratic Pythagorean Triples: Solution



Stereographic projection establishes a one-to-one correspondence between points on a circle and points on a line through it. Below are the formulas for the x-axis and the unit circle (which can be determined by characterizing the line through $(0,1)$ and (u,v) on the unit circle and $(x,0)$ on the x-axis, using point-slope form and different pairs of points for the slope):

$$(u,v) \mapsto \left(\frac{u}{1-v}, 0\right), \qquad \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right) \leftarrow\!\shortmid (x,0).$$

Since the formulas in both directions send rationals to rationals, this establishes a one-to-one correspondence between rational numbers and rational points on the unit circle. The pole $(0,1)$ corresponds to ∞ in the "extended" real number line $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, which one can imagines "wraps around."

Any Pythagorean triple (a,b,c) can be turned into a rational point $(\frac{a}{c}, \frac{b}{c})$ on the unit circle, and conversely any rational point $(\frac{x}{y}, \frac{w}{z})$ can be written using the lowest common denominator $(\frac{a}{c}, \frac{b}{c})$, which can be turned into a Pythagorean triple (a,b,c). Note (a,b,c) is primitive if and only if $(\frac{a}{c}, \frac{b}{c})$ has no smaller common denominator.

Stereographic projection sends the rational m/n to a rational point, which can in turn be turned into a primitive Pythagorean triple as follows:

$$\frac{m}{n} \mapsto \left(\frac{2mn}{m^2+n^2}, \frac{m^2-n^2}{m^2+n^2}\right) \mapsto (m^2-n^2, 2mn, m^2+n^2).$$

Note the formulas also work to turn rational functions $m(x)/n(x)$ into polynomial Pythagorean triples $(f_1(x), f_2(x), f_3(x))$ of the form

$$\begin{aligned} f_1(x) &= m(x)^2 - n(x)^2, \\ f_2(x) &= 2m(x)n(x), \\ f_3(x) &= m(x)^2 + n(x)^2. \end{aligned}$$

Again the statement about primitive triples and lowest terms holds, since sharing no common root is equivalent to sharing no common factor. If the second polynomial $f_2(x) = 2m(x)n(x)$ is quadratic, then either

- one of $m(x), n(x)$ is quadratic, the other constant; or
- both are linear, so that their product is quadratic.

If (f_1, f_2, f_3) is a quadratic triple then the former is impossible, since it would imply f_1 and f_3 have degree four (not quadratic), so we must have

$$\begin{aligned} m(x) &= Ax + B, \\ n(x) &= Cx + D. \end{aligned}$$

and therefore (f_1, f_2, f_3) has the form

$$\begin{aligned} f_1(x) &= (A^2 - C^2)x^2 + 2(AB - CD)x + (B^2 - D^2), \\ f_2(x) &= (2AC)x^2 + 2(AD + BC)x + (2BD), \\ f_3(x) &= (A^2 + C^2)x^2 + 2(AB + CD)x + (B^2 + D^2). \end{aligned}$$

The discriminants can then be calculated as

$$\begin{aligned} \Delta_1 &= [2(AB - CD)]^2 - 4(A^2 - C^2)(B^2 - D^2) \\ &= +4(AD - BC)^2, \\ \Delta_2 &= [2(AD + BC)]^2 - 4(2AC)(2BD) \\ &= +4(AD - BC)^2, \\ \Delta_3 &= [2(AC + BD)^2] - 4(A^2 + B^2)(C^2 + D^2) \\ &= -4(AD - BC)^2. \end{aligned}$$

Thus, by inspection, $\Delta_1 = \Delta_2 = -\Delta_3$. (Curiously, $AD - BC = \det[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$.)