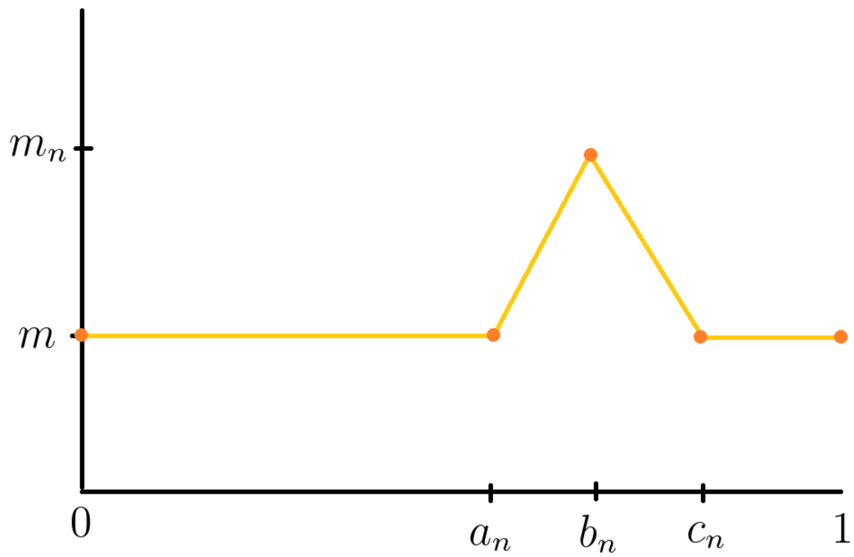
Pinching an Impulse: Solution

Pick any constant m, unbounded sequence $m_1, m_2, m_3, \cdots$ and three sequences (i) $a_1, a_2, a_3, \cdots$; (ii) $b_1, b_2, b_2, \cdots$; and (iii) $c_1, c_2, c_3, \cdots$ satisfying $0 < a_n < b_n < c_n < 1$ and $\lim_{n\to\infty} a_n = 1$. For example we could pick

$$m = 0, \quad m_n = n, \quad a_n = 1 - \frac{1}{n+1}, \quad b_n = 1 - \frac{1}{n+2}, \quad c_n = 1 - \frac{1}{n+3}.$$

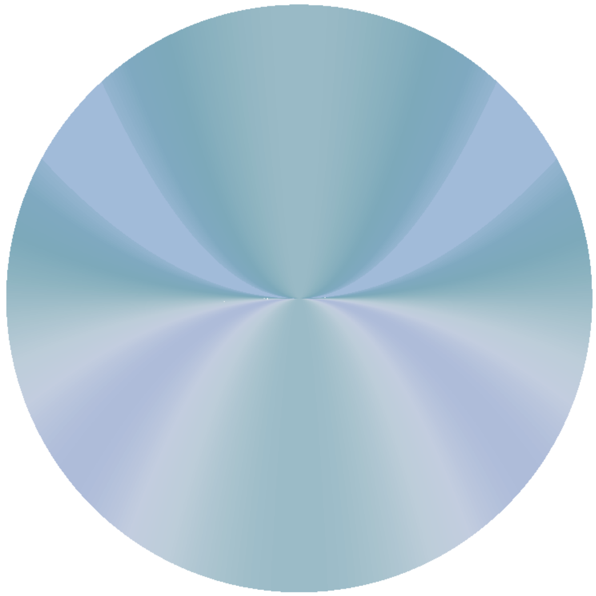
Then define $f_n(x)$ to be the piecewise-linear function whose graph consists of the line segments joining the points $(0, m)$, (a_n, m), (b_n, m_n), (c_n, m), $(1, m)$:



Since $f_n(1) = m$ for all n, so too does $f(1) = m$. For any $x < 1$ in the domain, eventually $a_n > x$ (or in other words, the triangle passes to the right of x), after which point $f_n(x) = m$, and thus $f(x) = m$ there too. Thus, f is the constant function identically equal to m, which is bounded.

This phenomenon shows up in some places.

Consider the continuity of $f(x, y) = 2x^2y/(x^4 + y^2)$ at $(0, 0)$. Approaching the origin along any line, the limit is 0. But approaching the origin along one of the parabolas $y = \pm x^2$, the limit is ± 1. This, and riffs on it, are common counterexamples given in introductory calculus classes.



Restricting the function f to a circle of radius r centered at the origin, the graph shows bumps on either side which, as $r \to 0$, get squeezed towards the East and West poles of the circle, never really reaching them.

Another illustration of this is seen in **Gibbs phenomenon**. Almost any "nice" function is expressible as an infinite sum of trigonometric functions, called a Fourier series. The partial sums of a Fourier series converge, yet there are "ringing artifacts" that are squeezed towards jump discontinuities.