# Icosian Palette: Solution

All six colors must be present on the dodecahedron. Once one half (one face and its five neighbors) has been painted, the other half is determined: every face must be painted the same color as the face on the opposite side. This is shown with front and back sides of an example below.

To see why this is true, suppose we start by painting one face our favorite color. The first ring of five neighboring faces must exhibit the five other colors. Call this the "front" side. The second ring of five faces beyond that, on the back side, cannot be colored the first color, or else one of the five faces in the first ring would be adjacent to two faces of the same color. This backside ring of five faces must exhibit five different colors, since they are all adjacent to the final face they surround and which is opposite the very first one. As the backside ring exhibits the same five colors as the frontside ring, that leaves only the first color for the final face opposite the very first one.


(front from front pov)


(back from front pov)


(back from back pov)

(The second picture above is what is seen if the front half is deleted, the third is what is seen if the solid is rotated 180° in 3D around the North-South axis.)

Thus, any painting of the dodecahedron is determined by the ring of colors around the face with our favorite color. There are 5! ways to put five colors to five faces, but we must take into account these are equivalent under rotation and reflection (from looking at the ring on the opposite side). There are five rotations and five reflections of a pentagon, ten total, so our answer is

$$\frac{5!}{10} = 12.$$

The title of the problem is based on the *Icosian Game*, a toy puzzle invented by mathematician William Rowan Hamilton in the late 1850s. The toy challenges the player to trace a circuit along a dodecahedron's edges which connects all the vertices, hitting each exactly once. Such a path is today called a Hamiltonian circuit. A general version of the problem was posed a couple years earlier by the graph theorist Kirkman.

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Hamilton himself considered the game as a way to apply what he called the icosian calculus, a way to describe the symmetries of the dodecahedron (which are the same as those of an icosahedron, because they are dual polytopes) in the language of group theory, aptly calling the symmetries "noncommutative roots of unity." His friend John T. Graves suggested turning the puzzle into a commercial venture (which apparently failed).