Killer Triangle Solution

In fact, we can show a stronger inequality:

$$60^\circ \le \frac{aA + bB + cC}{a\ +\ b\ +\ c} \le 90^\circ.$$

The upper bound is realized exactly when $\triangle ABC$ is degenerate (one-dimensional), and the lower bound is realized exactly when $\triangle ABC$ is equilateral.

First, we may replace 90° with $(A+B+C)/2$ and cross-multiply:

$$2aA + 2bB + 2cC \le (a+b+c)(A+B+C).$$

Expanding the right side (viewing $a+b+c$ as a single coefficient), then subtracting the terms from left to right and combining like terms yields

$$0 \le (b+c-a)A + (c+a-b)B + (a+b-c)C$$

The triangle inequality says each of these coefficients is positive (assuming it is nondegenerate). In other words, the shortest path between two points (in Euclidean space) is a straight line, which means the distance c on the triangle is shorter than the zig-zag distance $a+b$ the other way around the triangle.

As all of our operations were reversible, the truth of this last inequality implies the truth of the first one. That is, 90° is indeed the upper bound.

We can proceed in a similar fashion for the lower bound: replace 60° with $(A+B+C)/3$, and then cross-multiply to get the inequality

$$(a+b+c)(A+B+C) \le 3aA + 3bB + 3cC$$

Subtracting the left side from the right side, the right side then admits an elementary but nonetheless extremely non-obvious factorization:

$$0 \le (a-b)(A-B) + (b-c)(B-C) + (c-a)(C-A)$$

The triangle's angles and sides have the same order rankings (for example, if $A \ge B \ge C$ then $a \ge b \ge c$). Thus, every pair of factors above has the same sign (e.g. $a-b$ and $A-B$), and so all three products above are nonnegative. Indeed, the only way all of them are zero is if the triangle is equilateral.

As before, all the operations performed were reversible, so the truth of this last inequality implies the truth of the 60° lower bound.

This is an example of a **Coffin Problem**. These were examination problems given to Jewish candidates at Moscow State University during the 70s and 80s which had solutions that were wildly easier to understand than they were to discover (especially in a test setting), thus giving the mathematics department a means of discrimination with plausible deniability.

This particular problem is attributed to Podol'skii, Aliseichik, 1989 in the article *Entrance Examinations to the Mekh-mat* by A. Shen. Ilan Vardi has written a set of solutions to twenty of the problems listed in Shen's article:

`http://www.lix.polytechnique.fr/Labo/Ilan.Vardi/mekh-mat.html`

Tanya Khovanova also keeps an online collection of coffin problems:

`http://www.tanyakhovanova.com/coffins.html`

A similar-looking problem is Dranishnikov, Savchenko, 1984:

$$\frac{a+b-2c}{\sin C/2} + \frac{b+c-2a}{\sin A/2} + \frac{a+c-2b}{\sin B/2} \geq 0$$

which follows from rearranging the sum of three nonnegative terms

$$\sum_{\text{cyc}} (a-b)\left(\frac{1}{\sin B/2} - \frac{1}{\sin A/2}\right).$$

The notation $\sum_{\text{cyc}}$ (which is conventional, since expressions with these kinds of symmetry show up in certain contexts a lot) means to cycle through the letters alphabetically (wrapping back around as appropriate). For instance, the 60° lower bound earlier followed from rearranging $\sum_{\text{cyc}}(a-b)(A-B)$.