Halving Harmonics: Solution

Fill in the list $a_1, a_2, a_3, \cdots$ with powers-of-two for the non-square indices, and all other numbers in increasing order for square indices:

$$\sum_{k=1}^{n} \frac{1}{a_k} = {\color{red}1} + \frac{1}{2} + \frac{1}{4} + {\color{red}\frac{1}{3}} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + {\color{red}\frac{1}{5}} + \cdots$$

In the first n terms there are $\sqrt{n}$ (rounded) harmonic terms, and the others are geometric terms. The harmonic terms can be increased by decreasing the denominators to simply be $1, 2, 3, \cdots, \sqrt{n}$, so the harmonic part is bounded above by $H_{\sqrt{n}} < \ln \sqrt{n} + 1$. The geometric part is bounded above by the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$.

Therefore, for any choice $0 < \varepsilon < \frac{1}{2}$ (setting $c = \frac{1}{2} + \varepsilon$), we have

$$\sum_{k=1}^{n} \frac{1}{a_k} < \left[\ln \sqrt{n} + 1\right] + 1 < \left(\frac{1}{2} + \varepsilon\right) \ln n < c \sum_{k=1}^{n} \frac{1}{k}$$

for all $n > N$ so long as $1 + 1 < \varepsilon \ln N$.