Gyration Conjugation: Solution

An icosehdron has 12 vertices, 30 edges, and 20 triangular faces. If we mark a distinguished vertex then any rotational symmetry may be specified by first saying which vertex the marked one is sent to and then choosing one of the five rotated orientations around that final vertex, for a total of $12 \times 5 = 60$ rotations. A similar argument works for edges (30×2) or faces (20×3). This is an instance of the **orbit-stabilizer theorem** in group theory.

By symmetry, there are three types of axis a rotational symmetry of the icosehedron can have: one through a vertex or through the midpoint of either an edge or a face. Every vertex corresponds to five rotations, every edge to two, and every face to three. In the first case, for example, we have fifths of a full turn, or multiples of $360°/5 = 72°$. (Angles are measured counterclockwise according to the right-hand rule.)

A reflex-angle rotation around a ray is the same as a convex-angle rotation around the opposite ray (e.g. 270° around the South Pole is the same as 90° around the North Pole), so WLOG we may consider only convex angles. Then, not counting 0°, every vertex has two angles, every pair of opposite edges has one angle, and every face has one angle. In summary:

order	angle	count	$\times$	commute	$=$	total
1	0°	1	$\times$	60	$=$	60
2	180°	15	$\times$	4	$=$	60
3	120°	20	$\times$	3	$=$	60
5	72°	12	$\times$	5	$=$	60
5	144°	12	$\times$	5	$=$	60

As $AB = BA$ is equivalent to $ABA^{-1} = B$, we can count the (A, B) that commute by counting for each type of rotation A the rotations B unchanged by conjugation, tabulated above. In general, B simply must have the same axis of rotation, unless A is 180° around an edge midpoint and then B can also have a perpendicular axis (through another inscribed rectangle). Then

$$P = \frac{5 \times 60}{60 \times 60} = \frac{5}{60} = \frac{1}{12}.$$