Good Fibrations: Solution

Every neighboring pair of dodecahedra extends to a unique ring of ten dodecahedra, and every ring extends to a unique bundle of a dozen rings:

$$\text{pair} \to \text{ring} \to \text{bundle}$$

This is not a one-to-one correspondence, though: each bundle arises from any of its twelve rings, and similarly each ring arises from any of its ten neighboring pairs of dodecahedra.

To construct a neighboring pair of dodecahedra within a given ring, we can first pick one of the ten dodecahedra of the ring, then either of its two neighbors, but notice this overcounts by a factor of two since we can pick the two dodecahedra of a pair in two different orders - which is picked 1st vs 2nd.

Thus, there are $12 \times 10 = 120$ neighboring pairs per bundle.

To construct a neighboring pair in general, we can pick any of the 120 dodecahedra in the picture, then pick any of its 12 neighbors (a dodecahedron has twelve faces), and divide by 2 for the same reason as before.

Thus, there are $120 \times 12/2 = 720$ neighboring pairs in total.

Since there are 720 pairs total, and 120 pairs per bundle (and no pair shared between bundles), there must be $720/120 = 6$ bundles.

This counting argument also works in the game SET.

In SET, each of the cards has a picture with four features (color, shape, number, shading), each with three possible variations, for a total of $3^4 = 81$:

- color: red, purple, green
- shape: oval, squiggly, diamond
- number: one, two, three
- shading: blank, solid, hatching

A "SET" is three cards in which each feature either has the same variation on each card or all three variations. We can write down the equation

$$(\text{SETs}) \cdot (\text{pairs per SET}) = (\text{pairs}) \cdot (\text{SETS per pair})$$

There are $\binom{3}{2} = 3$ pairs of cards per SET, and there is 1 SET per pair (in any SET, the features of the third card are determined by those of the first two). And the total number of pairs is $\binom{3^4}{2}$, so the number of SETs is

$$\binom{3^4}{2} \Big/ \binom{3}{2} = 1080.$$

For dodecahedral bundles we used the same reasoning, with adjacent pairs of dodecahedra instead of pairs of cards and bundles of rings instead of SETs!

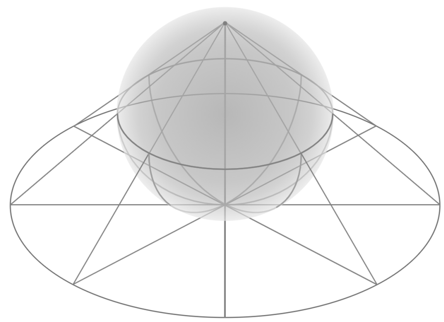
SET is an example of a **Steiner system**. A system $S(t, k, n)$ is a collection of k-subsets (called blocks) of an n-set for which every t-subset is contained within exactly one block. By our counting argument, there are $\binom{n-\ell}{t-\ell} / \binom{k-\ell}{t-\ell}$ blocks containing any ℓ-subset. SET is a $S(2, 3, 3^4)$ and $\ell = 0$ counts SETs.

There are infinitely-many lines (not necessarily through the origin) in Euclidean space. If we consider 4D space, and instead of using real numbers for coordinates use the integers mod 3, then the vectors and lines respectively correspond to cards and SETs from the game SET!

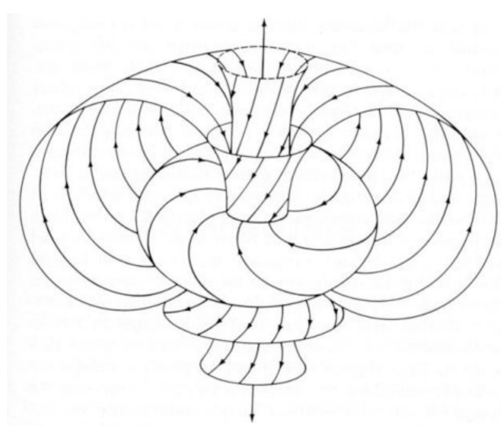
This problem's title and bundle picture are taken from a post of the same name on the blog "Complex Projective 4-Space."

The dodecahedral bundles are discrete versions of the **Hopf fibration**.

Visualizing the fibration requires stereographic projection. Usually, we project a circle onto a line, or a sphere onto a plane, but for this, we need to project the “three-sphere” sitting in 4D down to 3D Euclidean space.



Just as a Möbius band is a bunch of line segments arranged in a circle, or a Klein bottle is a bunch of circles arranged in a circle, the three-sphere is, somewhat miraculously, a bunch of circles arranged in the shape of a (2D) sphere! When stereographically projected, that means all of 3D space is filled in with circles, with one “infinitely large” circle (i.e. a line).



The circles can be bunched together into wreaths (solid Dupin cyclides, to be exact), then those wreathes turned into rings of dodecahedra.

In 4D space, these dodecahedra are the cellular panels of the “120-cell” polytope. The centers of the dodecehdra form the dual polytope, the “600-cell,” which is also the group of unit-length **icosians** in the quaternions. Because of how quaternions model 3D rotations, every antipodal pair of icosians corresponds to one of the 60 rotational symmetries of an icosahedron!