Favorite Angle: Solution

Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are (WLOG) unit vectors at 120° angles to each other.

Remember the dot product is bilinear and $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$. That means each of the three dot products $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{b} \cdot \mathbf{c}$, $\mathbf{c} \cdot \mathbf{a}$ is equal to $-\frac{1}{2}$.

The most "nose-to-the-ground" solution solves for $\mathbf{c}$ as a linear combination of $\mathbf{a}, \mathbf{b}$ using this information. We can pick a unit normal vector $\mathbf{n}$ to the plane spanned by $\{\mathbf{a}, \mathbf{b}\}$, then $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ is a basis and $\mathbf{c} = u\mathbf{a} + v\mathbf{b} + w\mathbf{n}$ for some coefficients u, v, w. The equations $\mathbf{a} \cdot \mathbf{c} = -\frac{1}{2}$ and $\mathbf{b} \cdot \mathbf{c} = -\frac{1}{2}$ become

$$\begin{cases} \quad u - \frac{1}{2}v = -\frac{1}{2} \\ -\frac{1}{2}u + \quad v = -\frac{1}{2} \end{cases} \implies \begin{cases} u = -1 \\ v = -1 \end{cases}$$

and then $1 = \|\mathbf{c}\|^2 = u^2 - uv + v^2 + w^2$ becomes $1 = 1 + w^2$ which forces $w = 0$, thus $\mathbf{c} = -\mathbf{a} - \mathbf{b}$ and we conclude $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly dependent.

This implies the identity $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, which in turn suggests there might be a simple solution involving symmetry. Indeed, we can just distribute

$$\begin{aligned} \|\mathbf{a} + \mathbf{b} + \mathbf{c}\|^2 &= (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) \\ &= 3(1) + 6(-\tfrac{1}{2}) \ = \ 0. \end{aligned}$$

and immediately conclude from this that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$.

A similar trick can be used to show the vertices of regular tetrahedron, centered at the origin, are at $\cos^{-1}(-\frac{1}{3})$ angles to each other. Note this proof showing three vectors at 120° must be coplanar actually works in any number of dimensions, not just 3D. (The previous solution can also be made to work in n dimensions, by decomposing $\mathbf{c} = \mathbf{c}_\parallel + \mathbf{c}_\perp$ into parallel and perpendicular components WRT $\mathrm{span}\{\mathbf{a}, \mathbf{b}\}$, then $\mathbf{c}_\perp$ takes the role of $\mathbf{n}$.)

The volume of the parallelepiped generated by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is the triple product

$$\mathrm{vol} = \det(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

If we define the matrix $V = (\mathbf{a}\ \mathbf{b}\ \mathbf{c})$ (assuming the three vectors are column vectors), we can use the so-called **Gramian determinant**

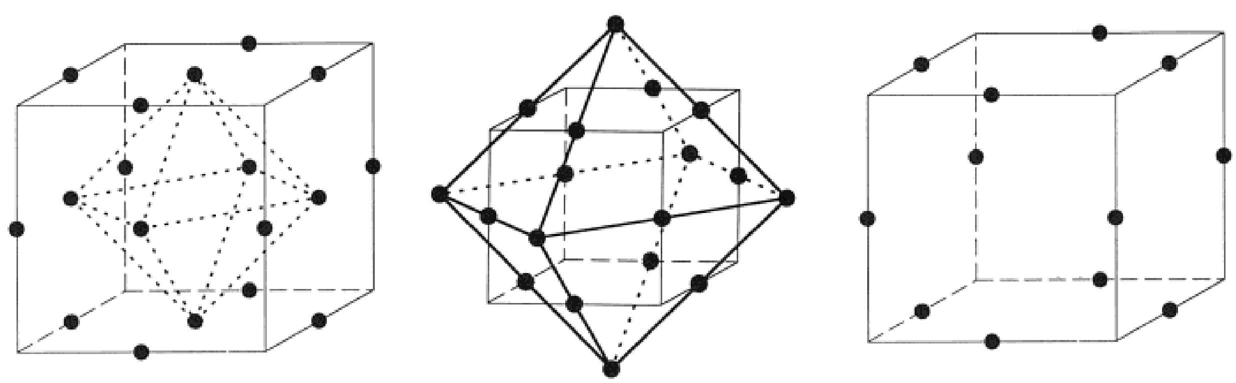
$$(\det V)^2 = (\det V)(\det V) = (\det V^T)(\det V) = \det(V^T V)$$
$$= \det \begin{pmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{pmatrix} = \det \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} = 0.$$

But $\det V = 0$ implies the columns of $V = (\mathbf{a}\ \mathbf{b}\ \mathbf{c})$ are linearly dependent!

(In fact, $\det(V^T V) = \mathrm{vol}^2$ applies for the parallelepiped generated by any number of vectors in any number of dimensions. The expression $\det V$ by itself doesn't make sense unless V is a square matrix.)

Note the last two solutions begin with the serendipitous decision of squaring, after which the algebra works out. This is a common trick in some circles, a special case of two mutually inverse tricks, polarization and symmetrization: loosely speaking, these convert between bilinear and quadratic gadgets.

A **root system** Φ is a particularly symmetric set of vectors (called roots); any line through a root contains only one other, its antipode; reflecting one root across the plane perpendicular to a second root gives a third root; projecting one root onto another produces an integer or half-integer multiple.



By adding complex numbers into the mix, root systems can be used to classify (infinitesimal versions of) *smooth* symmetries. To compare smooth vs. discrete symmetry, consider the symmetry of a sphere vs. of a polyhedron.

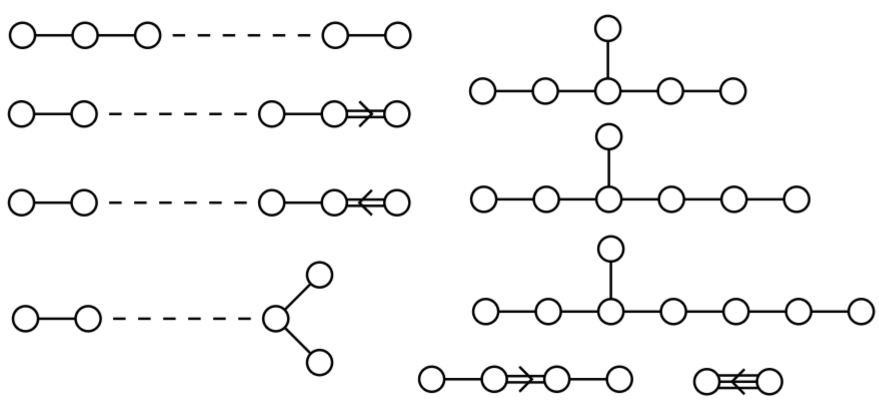
The angles between roots are quite restricted - they are among the special angles studied in school trigonometry, the 30° and 45° families. The ratio between lengths of non-orthogonal roots can only be one of $\sqrt{1}, \sqrt{2}, \sqrt{3}$.

This is also related to the **crystallographic restriction theorem**, which validates the empirical observation that crystals in nature have twofold, threefold, fourfold, or sixfold rotational symmetry and no other kind.

Picking a plane separates a root system Φ into "positive" and "negative" halves, $\Phi = \Phi^+ \cup \Phi^-$. A basis $\Delta \subseteq \Phi^+$ of "simple" roots can be chosen so that all positive roots are sums of simple ones. All but at most one pair of simple roots are either orthogonal or at 120° angles to each other.

From Δ we can construct a **Dynkin diagram**: a graph, consisting of one node for each simple root and a simple edge for each pair at 120° angles (the only possible exceptions are directed double or triple edges for the supplements of 45° or 30° respectively, pointing from larger to shorter root).

It is possible to reconstruct root systems from their Dynkin diagrams. And more generally, Dynkin diagrams are used to classify many kinds of interrelated mathematical objects involving symmetry and geometry.



The Dynkin diagrams for "irreducible" root systems are classified.