Factorial Frenzy: Solution

Expanding $(1+x)^{m+n}(1-x)^{m+n} = (1-x^2)^{m+n}$, the x^{2m} coefficient is

$$\sum_k \binom{m+n}{m+k}\binom{m+n}{m-k}(-1)^{m-k} = \binom{m+n}{m}(-1)^m.$$

To understand the left side, note each x^{2m} term arises from multiplying a $\binom{m+n}{m+k}x^{m+k}$ term from $(1+x)^{m+n}$ and a $\binom{m+n}{m-k}(-x)^{m-k}$ term from $(1-x)^{m+n}$ together for some k. The right side is just the coefficient of $\binom{m+n}{m}(-x^2)^m$.

Rewriting the binomial coefficients with factorials gives

$$\sum_k \frac{(m+n)!}{(m+k)!(n-k)!}\cdot\frac{(m+n)!}{(m-k)!(n+k)!}(-1)^{m-k} = \frac{(m+n)!}{m!n!}(-1)^m.$$

The $(-1)^m$ can be cancelled from both sides, and $(-1)^{-k}$ might as well be written $(-1)^k$. There is an abundance of $(m+n)!$s, so let's divide by both of them from the LHS numerator to end up with one in the RHS denominator. Furthermore, the $m \pm k$ and $n \pm k$ in the LHS denominator sum to $2m$ and $2n$ respectively, so let's multiply both sides by $(2m)!(2n)!$. We get

$$\sum_k \frac{(2m)!}{(m+k)!(m-k)!}\frac{(2n)!}{(n-k)!(n+k)!}(-1)^k = \frac{(2m)!(2n)!}{m!n!(m+n)!}.$$

The RHS can be "unsimplified" in preparation to write binomial coefficients by multiplying numerator and denominators by $m!$ and $n!$:

$$\frac{(2m)!(2n)!}{m!n!(m+n)!} = \frac{(2m)!}{m!m!}\cdot\frac{(2n)!}{n!n!}\cdot\frac{m!n!}{(m+n)!} = \binom{2m}{m}\binom{2n}{n}\Big/\binom{m+n}{m}$$

Thus we conclude the **von Szily convolution identity**:

$$\sum_k (-1)^k \binom{2m}{m+k}\binom{2n}{n-k} = \binom{2m}{m}\binom{2n}{n}\Big/\binom{m+n}{m}$$

Ira Gessel called $S(m,n) = \binom{2m}{m}\binom{2n}{n}/\binom{m+n}{m}$ the **super Catalan numbers**.

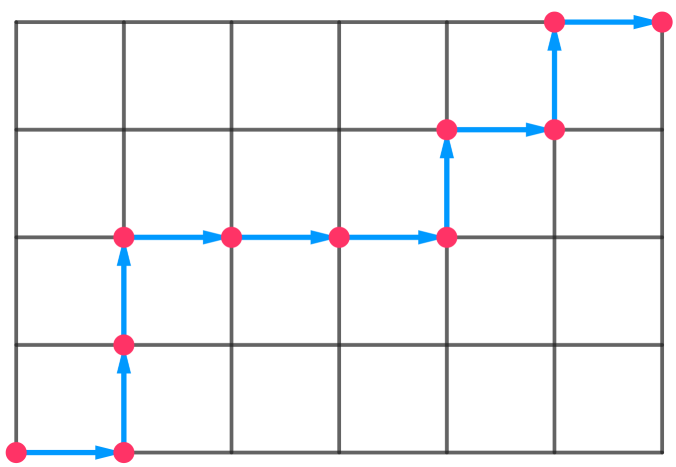
It is not obvious from the rational expression that these should even be whole numbers, let alone (surprisingly) what they ought to count!

A couple alternative ways to show they are whole numbers involve:

- **Legendre's formula**: the exponent of a prime p in the factorization of $n!$ is $(n-s)/(p-1)$, where s is the sum of n's digits in base p.
- **Kummer's theorem**: the exponent of a prime p in the factorization of $\binom{n}{k}$ is the number of carries when adding k and $(n-k)$ in base p.

It is possible to give $S(m,n)$ a combinatorial interpretation via von Szily.

A **lattice path** is a sequence of unit steps north and east from one grid point (usually the origin) to another in the Euclidean plane. The number of lattice paths from $(0,0)$ to (m,n) is $\binom{m+n}{m}$, because a lattice path corresponds to a choice of which m of the $m+n$ steps go right and which n go up.



Consider lattice paths $(0,0) \to (m+n, m+n)$. There are $\binom{2m+2n}{m+n}$ total. Any one of them must intersect the line $x+y=2m$ at some point $(m+k, m-k)$ which is k squares diagonally from (m,m) for some k. (Note this argument also works using $x+y=2n$ instead.) There are $\binom{2m}{m+k}$ paths $(0,0) \to (m+k, m-k)$ and $\binom{2n}{n-k}$ paths $(m+k, m-k) \to (m+n, m+n)$.

Thus, let E and O be the number of paths $(0,0) \to (m+n, m+n)$ which intersect $x+y=2m$ at a point an even or odd number of squares from (m,m). Then super Catalan numbers are the difference $S(m,n) = E - O$.