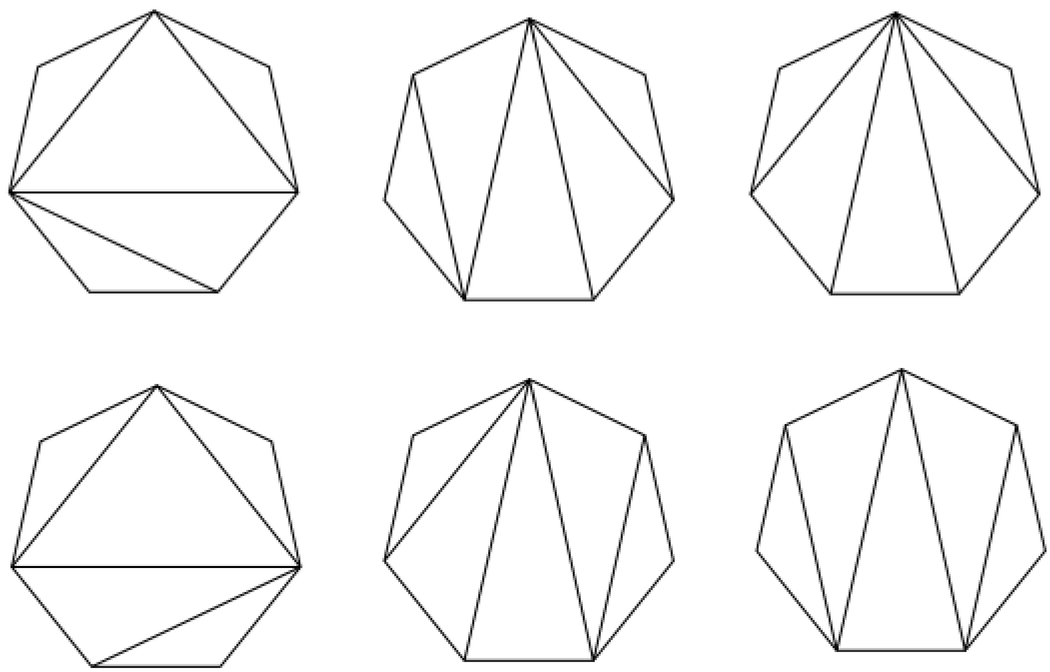
Cyclic Sieving: Solution



Often, combinatorial formulas counting certain things with set-theoretic descriptions generalize to other formulas (their q-**analogs**) which count similar things with linear-algebra interpretations. (Arguably it would be more truthful to say projective-geometry interpretations.) The analogs use the traditional choice of variable q. Here, linear algebra is not done over the fields $\mathbb{R}$ or $\mathbb{C}$, but rather over a **finite field** with q scalars denoted $\mathbb{F}_q$.

The simplest example: n counts how many elements the set $\{1, \cdots, n\}$ has, while $[n] := q^{n-1} + \cdots + q + 1$ counts how many 1D subspaces the n-dimensional vector space $\mathbb{F}_q^n$ has. More generally, the binomial coefficient $\binom{n}{k} = \frac{n(n-1)\cdots}{k(k-1)\cdots}$ (with k terms in the numerator and denominator) counts how many size-k subsets there are of $\{1, \cdots, n\}$, and its q-analog $\left[{n \atop k} \right] = \frac{[n][n-1]\cdots}{[k][k-1]\cdots}$ counts how many k-dimensional subspaces there are of the vector space $\mathbb{F}_q^n$.

Plugging $q = 1$ into q-analogs typically gives the original combinatorial formula. This suggests a "**field with one element**" is missing in field theory, however changing the definition of a "field" to allow only one element fails to reproduce the combinatorial formulas. Mathematicians have tried to remedy this by abstracting every possibly relevant definition until they can finally actually define $\mathbb{F}_1$, but this saga has yet to reach a conclusion.

The **cyclic sieving phenomenon** (CSP) occurs when a combinatorial formula counts certain things and then plugging a complex nth root of unity into the q-analog counts how many of those things have cyclic symmetry.

For instance, the formula $\binom{n}{k}$ counts how many ways there are to color k vertices of a polygon one color and the other $n-k$ vertices another (like a necklace with beads), and if d is a factor of n then plugging a (primitive) dth root of unity for q into the analog $\left[{n \atop k}\right]$ tells us how many of those colorings are unchanged by rotating the polygon by $1/d$th of a full turn.

In our problem of polygon triangulations, however, we are not counting the "fixed points" of rotations (the configurations with cyclic symmetry) but rather the "orbits," or in other words we are grouping the triangulations according to rotations and then counting how many groups (orbits) we get. But we can count orbits using fixed points according to **Burnside's Lemma**.

(If we did the same for coloring vertices of a polygon, grouping the colorings according to rotations, **necklace polynomials** count the orbits.)

The total number of triangulations of a polygon with n vertices, where rotations are *not* counted as equivalent, is C_{n-2} where the **Catalan numbers** are given by the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$. The q-analog $\frac{1}{[n+1]}\left[{2n \atop n}\right]$ exhibits CSP, so plugging a complex (primitive) dth root of unity for q in (where d is a factor of n) yields how many configurations are unchanged by $1/d$th of a full turn, which Burnside's Lemma tells us how to count orbits with.