Categorical Imperative: Solution

First, we can set $a, d = \blacklozenge$ and $c, d = \lozenge$. In our 2D representation, we can either evaluate the expression row-wise to get $\blacklozenge \bullet \blacklozenge$ (which simplifies to $\blacklozenge$), or evaluate the expression column-wise to get $\lozenge \circ \lozenge$ (which simplifies to $\lozenge$):

$$\begin{array}{ccccc} \blacklozenge & \circ & \lozenge & & \blacklozenge \\ \bullet & & \bullet & \rightarrow & \bullet \\ \lozenge & \circ & \blacklozenge & & \blacklozenge \\ & \downarrow & & & \\ \lozenge & \circ & \lozenge & & \end{array}$$

To emphasize $\lozenge$ and $\blacklozenge$ are the same element, and an identity element for both operations $\circ$ and $\bullet$, we will simply call it ${\color{magenta}\blacklozenge}$. Next, we can evaluate

$$\begin{array}{ccc} a & \circ & {\color{magenta}\blacklozenge} \\ \bullet & & \bullet \\ {\color{magenta}\blacklozenge} & \circ & b \end{array}$$

either row-wise to get $a \bullet b$ or column-wise to get $a \circ b$. Thus, $a \bullet b = a \circ b$ are the same operation! For this reason, we will now use ${\color{cyan}\bullet}$ instead of $\circ$ or $\bullet$.

Finally, we can evaluate one last expression

$$\begin{array}{ccc} {\color{magenta}\blacklozenge} & {\color{cyan}\bullet} & a \\ {\color{cyan}\bullet} & & {\color{cyan}\bullet} \\ b & {\color{cyan}\bullet} & {\color{magenta}\blacklozenge} \end{array}$$

either row-wise to get $a \,{\color{cyan}\bullet}\, b$ or column-wise to get $b \,{\color{cyan}\bullet}\, a$. Thus, $a \,{\color{cyan}\bullet}\, b = b \,{\color{cyan}\bullet}\, a$, meaning the operation is commutative.

This line of reasoning, called the **Eckman-Hilton argument**, is a style of 2D diagrammatic proof which can be used, for example, to show higher homotopy groups are trivial. An approximately accurate version of this statement in plainer, albeit vaguer, language might read: conjoining together higher-dimensional holes in topological spaces is a commutative operation.

The **interchange law** $(a \circ b) \bullet (c \circ d) = (a \bullet c) \circ (b \bullet d)$ shows up naturally in the context of monoidal 2-categories. But what is a category?

A category is a collection of objects and composable arrows. The most well-known example is the category of sets: the objects are just sets, the arrows are functions. There are categories of topological spaces with continuous functions, lattices with monotone functions, groups with homomorphisms, and categories for numerous other kinds of mathematical objects.

Categories are monoidal when they have an operation to combine objects together (which in turn combines arrows together). A 2-category has not only objects and composable arrows between objects, but composable arrows between those arrows! For instance, a path on a surface may be considered an arrow between endpoints, and then there are homotopies - ways of sliding paths across the surface to turn them into other paths.