Ensemble Cast: Solution

The characteristic polynomial of the symmetric matrix H is

$$\det\begin{pmatrix} x+y-\lambda & z \\ z & x-y-\lambda \end{pmatrix} = (x-\lambda)^2 - y^2 - z^2.$$

The eigenvalues ($\lambda_1 \le \lambda_2$) of H are therefore given by

$$\begin{cases} \lambda_1 = x - \sqrt{y^2+z^2} \\ \lambda_2 = x + \sqrt{y^2+z^2} \end{cases}$$

Thus, the (x, y, z) corresponding to a given (λ_1, λ_2) satisfy

$$\begin{cases} x = \frac{1}{2}(\lambda_1 + \lambda_2) \\ y^2 + z^2 = \frac{1}{4}(\lambda_2 - \lambda_1)^2 \end{cases}$$

This is a circle in xyz-space. The density ρ of H at every point on it is

$$\begin{aligned} \rho &= \pi^{-3/2} \exp\big(-(x^2+y^2+z^2)\big) \\ &= \pi^{-3/2} \exp\big(-\tfrac{1}{4}(\lambda_1+\lambda_2)^2 - \tfrac{1}{4}(\lambda_2-\lambda_1)^2\big) \\ &= \pi^{-3/2} \exp\big(-\tfrac{1}{2}(\lambda_1^2+\lambda_2^2)\big) \end{aligned}$$

Thus, the density ρ is constant on the circle associated to a given (λ_1, λ_2).

The circumference of the circle is $2\pi\sqrt{y^2+z^2} = \pi|\lambda_2 - \lambda_1|$. Since ρ is constant on the circle, our answer is simply ρ times this circumference:

$$\frac{1}{\sqrt{\pi}} \exp\left(-\frac{\lambda_1^2+\lambda_2^2}{2}\right)|\lambda_2 - \lambda_1|$$

This is a Gaussian Orthogonal Ensemble (GOE). Using complex Hermitian matrices instead of real symmetric ones gives the **Gaussian Unitary Ensemble** (GUE), proposed by theoretical physicist Eugene Wigner as a way to model the spectral theory (energy levels) of heavy atomic nuclei.

Notice $\rho = 0$ on the line $\lambda_1 = \lambda_2$. As a result, (λ_1, λ_2) exhibits **repulsion**: the eigenvalues are not independent, they prefer to be apart from each other.