Blinding Sphere: Solution

We wish to integrate the light intensity $\|\mathbf{r}-\mathbf{p}\|^{-2}$ as $\mathbf{r}$ varies over the surface of the unit sphere (centered at $\mathbf{0}$). Spherical coordinates are given by

$$\mathbf{r}(\phi,\theta) = \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix}$$

This is the math convention, with ϕ the polar angle and θ the azimuthal angle; the physics convention is reversed. By symmetry, we can pick $\mathbf{p} = (0, 0, p)$.

For the integral, our surface area element is given by

$$\left\|\frac{\partial\mathbf{r}}{\partial\phi} \times \frac{\partial\mathbf{r}}{\partial\theta}\right\| = \sin\phi.$$

Chugging through the calculations we journey forth:

$$\frac{1}{\operatorname{area}(S)}\int_S \frac{\mathrm{d}A}{\|\mathbf{r}-\mathbf{p}\|^2} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta}{(\cos\phi-p)^2+\sin^2\phi}$$

$$= \frac{2\pi}{4\pi}\int_0^{\pi} \frac{-\mathrm{d}(\cos\phi)}{(\cos\phi-p)^2+1-\cos^2\phi} = \frac{1}{2}\int_{-1}^{1}\frac{\mathrm{d}t}{(t-p)^2+1-t^2}$$

$$= \frac{1}{2}\int_{-1}^{1}\frac{\mathrm{d}t}{(1+p^2)-(2p)t} = \frac{1}{2}\left[-\frac{1}{2p}\ln\Big((1+p^2)-(2p)t\Big)\right]_{-1}^{1}$$

$$= -\frac{1}{4p}\ln\left(\frac{1+p^2-2p}{1+p^2+2p}\right) = \frac{1}{2p}\ln\left|\frac{1+p}{1-p}\right| = \frac{1}{2p}\ln\left|\coth\left(\frac{1}{2}\ln p\right)\right|.$$

The last two expressions are both valid answers, whether $0 < p < 1$ or $p > 1$.