Bipolarity: Solution

Law of Cosines says $2^2 = d_1^2 + d_2^2 - 2d_1d_2\cos\sigma$.

Law of Sines says $\dfrac{\sin\sigma}{2} = \dfrac{(y/d_1)}{d_2} = \dfrac{(y/d_2)}{d_1}$.

Therefore $d_1d_2\cos\sigma = r^2 - 1$ and $d_1d_2\sin\sigma = 2y$. Plus, we have

$$\begin{cases} d_1^2 = (x+1)^2 + y^2 \\ d_2^2 = (x-1)^2 + y^2 \end{cases} \implies \begin{cases} d_1^2 + d_2^2 = 2(r^2+1) \\ d_1^2 - d_2^2 = 4x \end{cases}$$

Now compute $f(\tau)/g(\sigma)$ as follows:

$$\begin{aligned}
\frac{\cosh\tau}{\cos\sigma} &= \frac{\left(\dfrac{d_1}{d_2} + \dfrac{d_2}{d_1}\right)}{2\cos\sigma} &= \frac{d_1^2 + d_2^2}{2d_1d_2\cos\sigma} &= \frac{r^2+1}{r^2-1}, \\
\frac{\sinh\tau}{\cos\sigma} &= \frac{\left(\dfrac{d_1}{d_2} - \dfrac{d_2}{d_1}\right)}{2\cos\sigma} &= \frac{d_1^2 - d_2^2}{2d_1d_2\cos\sigma} &= \frac{2x}{r^2-1}, \\
\frac{\cosh\tau}{\sin\sigma} &= \frac{\left(\dfrac{d_1}{d_2} + \dfrac{d_2}{d_1}\right)}{2\sin\sigma} &= \frac{d_1^2 + d_2^2}{2d_1d_2\sin\sigma} &= \frac{r^2+1}{2y}, \\
\frac{\sinh\tau}{\sin\sigma} &= \frac{\left(\dfrac{d_1}{d_2} - \dfrac{d_2}{d_1}\right)}{2\sin\sigma} &= \frac{d_1^2 - d_2^2}{2d_1d_2\sin\sigma} &= \frac{x}{y}.
\end{aligned}$$

We also get $\tan\sigma = \dfrac{2y}{r^2-1}$ and $\tanh\tau = \dfrac{2x}{r^2+1}$ from these. Now

$$\begin{aligned}
\frac{\cosh\tau}{\sinh\tau} - \frac{\cos\sigma}{\sinh\tau} &= \frac{r^2+1}{2x} - \frac{r^2-1}{2x} &= \frac{1}{x} \\
\frac{\cosh\tau}{\sin\sigma} - \frac{\cos\sigma}{\sin\sigma} &= \frac{r^2+1}{2y} - \frac{r^2-1}{2y} &= \frac{1}{y}
\end{aligned}$$

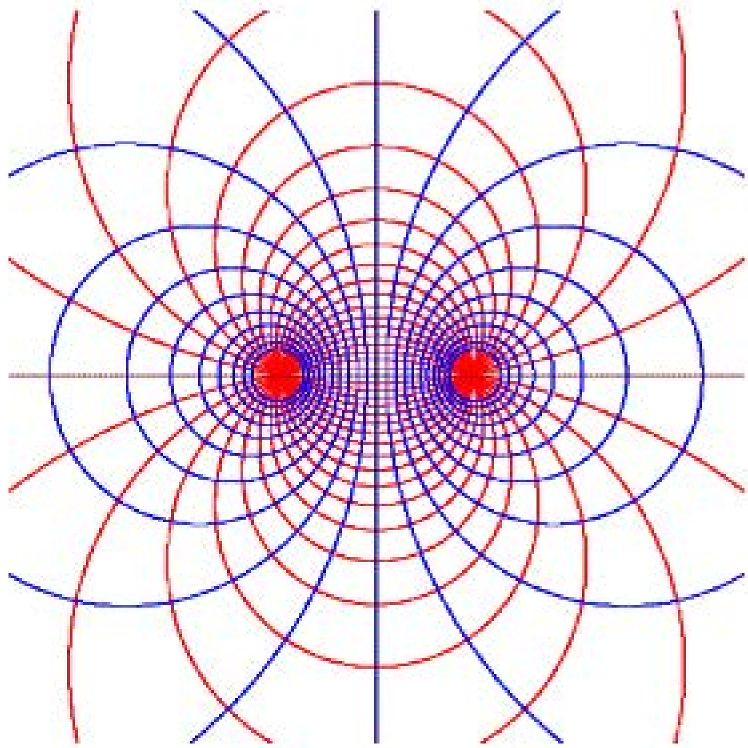
Therefore, $(x,y) = \left(\dfrac{\sinh\tau}{\cosh\tau - \cos\sigma}, \dfrac{\sin\sigma}{\cosh\tau - \cos\sigma}\right)$.

This gives rise to **bipolar coordinates** as follows.

First complete the square from the formulas for $\tan\sigma$ and $\tanh\tau$:

$\tan\sigma = \dfrac{2y}{r^2-1}$	$\tanh\tau = \dfrac{2x}{r^2+1}$
$x^2+y^2-2(\cot\sigma)y = 1$	$x^2-2(\coth\tau)x+y^2 = -1$
$x^2+(y-\cot\sigma)^2 = \csc^2\sigma$	$(x-\coth\tau)^2+y^2 = \operatorname{csch}^2\tau$

The circles of constant σ (red) and constant ρ (blue) are below:



This is what we get when we stereographically project longitude and latitude coordinates on a sphere from a point on its equator to a plane!

These are **circles of Apollonius**. The red and blue circles are orthogonal families of circles, and every blue circle is the locus of points with a constant ratio of distances to the two foci (aka poles), namely $d_1/d_2 = \exp\tau$.