Arts and Crafts: Solution

For a given thread (associated to a value t), we may calculate its slope

$$m = \frac{y-t}{x-0} = \frac{0-t}{(1-t)-0}$$

Solving for y yields the formula

$$y \;=\; t - \frac{tx}{t-1} \;=\; t + x + \frac{x}{t-1}.$$

Using Algebra. We may see for which points (x, y) (in the unit square, $0 \leq x, y \leq 1$) there exists a solution t (in the interval $0 \leq t \leq 1$). Multiplying by $(t-1)$ and rearranging gives $t^2 + (x - y + 1)t + y = 0$, with solution

$$t = \frac{(y-x+1) \pm \sqrt{(y-x+1)^2 - 4y}}{2}$$

Denote $v = y - x + 1$ and $\Delta = v^2 - 4y$. Since $0 \leq x, y \leq 1$, we know $v \geq 0$. Suppose $\Delta \geq 0$. Then, since $\Delta \leq v^2$, we know $\sqrt{\Delta} \leq v$, and therefore

$$0 \;\leq\; \frac{v - \sqrt{\Delta}}{2} \;\leq\; \frac{v}{2} \;\leq\; \frac{1-0+1}{2} \;=\; 1.$$

That is, as long as $0 \leq x, y \leq 1$ and $\Delta \geq 0$, there is a solution $t = (v - \sqrt{\Delta})/2$ in the interval $0 \leq t \leq 1$. In other words, the point (x, y) is on or below some thread. Thus, the curve is defined by the boundary of this inequality, $\Delta = 0$.

Square rooting $v^2 = 4y$ yields $y - x + 1 = 2\sqrt{y}$, then subtracting $2\sqrt{y}$ and adding x, we can factor as $(\sqrt{y} - 1)^2 = x$. Since $y \leq 1$, square rooting again yields $\sqrt{y} - 1 = -\sqrt{x}$. In conclusion, the curve is the so-called **superellipse**

$$\sqrt{x} + \sqrt{y} = 1.$$

Parabola. The curve opens up diagonally, so we ought to see what happens if we rotate our coordinate system by 45°. Let's introduce

$$u = \frac{x-y}{\sqrt{2}}, \quad v = \frac{x+y}{\sqrt{2}},$$

the equation $(y - x + 1)^2 = 4y$ becomes $(1 - \sqrt{2}u)^2 = 2\sqrt{2}(v - u)$, and then completing the square gets us $\sqrt{2}v = \frac{1}{2} + u^2$. Thus, the curve is a parabola.

With Calculus. Calculate the derivative of y with respect to t:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - \frac{x}{(t-1)^2}.$$

The critical point (where the y coordinate is maximized, which is on the curve) occurs where $\mathrm{d}y/\mathrm{d}t = 0$, or in other words $(t-1)^2 = x$. Since we want $0 \le t \le 1$, this yields the solution $t = 1 - \sqrt{x}$, and substituting back in and factoring yields $y = (1 - \sqrt{x})^2$, or once again $\sqrt{x} + \sqrt{y} = 1$.