Problem of the week #9: Solution

Part (a). A function of the form $f(x) = \frac{ax+b}{cx+d}$ is called a linear fractional transformation, or a Möbius transformation. Its derivative has numerator $\delta = ad - bc$. Notice this is the determinant of $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$!

The first three derivatives of $y = (ax+b)/(cx+d)$ are

$$\begin{aligned} y' &= \frac{\delta}{(cx+d)^2}, \\ y'' &= \frac{-2c\delta}{(cx+d)^3}, \\ y''' &= \frac{6c^2\delta}{(cx+d)^4}. \end{aligned}$$

The three exponents are related by $4 + 2 = 3 + 3$, so find

$$\begin{aligned} y'y''' &= \frac{6c^2\delta^2}{(cx+d)^6}, \\ y''y'' &= \frac{4c^2\delta^2}{(cx+d)^6}. \end{aligned}$$

They differ only by their coefficients, so we may conclude

$$2y'y''' = 3\big(y''\big)^2.$$

Conversely, we should show this has no other solutions.

Given the above differential equation, we can first address when $y' \equiv 0$ or $y'' \equiv 0$ identically: in this case $y = ax + b$.

Otherwise, we may address an interval where $y'' \neq 0$, and hence (from the differential equation) $y' \neq 0$ too, and futhermore we may assume it is positive, $y' > 0$, by replacing y with $-y$ if necessary.

First, divide to put in logarithmic derivative form $\big(\ln|f|\big)' = f'/f$,

$$2\,\frac{y'''}{y''} = 3\,\frac{y''}{y'}$$

so that both sides integrate to become logarithms:

$$2\ln|y''| = 3\ln|y'| + C$$

Exponentiate, drop the absolute values by introducing a $\pm$ sign, then replace e^C with C (which absorbs the $\pm$ sign). We get

$$\big(y''\big)^2 = C\big(y'\big)^3.$$

Since $(y'')^2 > 0$ and $y' > 0$, it follows $C > 0$ and we may take square roots. Doing this, replace $\sqrt{C}$ with C (again absorbing an implicit $\pm$):

$$\big(y'\big)^{-3/2} y'' = C$$

Integrating (and dividing by -2, absorbing into constants) gives

$$\big(y'\big)^{-1/2} = Cx + D$$

Then isolate y' to get

$$y' = \frac{1}{(Cx+D)^2}$$

Integrating again gives

$$y = -\frac{1/C}{Cx+D} + B$$

which, when combined, is of the form $y = \dfrac{ax+b}{cx+d}$. On any interval where this is defined, y' and y'' cannot be 0, as assumed.

Note there are only three degrees of freedom to this form, despite there being four unknowns a, b, c, d. This is because multiplying all four by a value doesn't change the function (without loss of generality, we may assume $ad - bc = \pm 1$, which is done in some contexts).

Part (b). The first three terms in $f(x)$s Taylor series around $x = w$:

$$f(x) \approx f(w) + f'(w)(x-w) + \frac{f''(w)}{2}(x-w)^2$$

If $f'(w) = 0$, the only Mobius transformation with second derivative 0 are constant functions, so the best approximation is the constant $f(w)$.

Otherwise, we may subtract $f(w)$ and divide by $f'(w)(x-w)$ to get

$$\frac{1}{f'(w)}\frac{f(x)-f(w)}{x-w} \approx 1 + \frac{f''(w)}{2f'(w)}(x-w)$$

The right side is the first two terms of a geometric series,

$$1 + \frac{f''(w)}{2f'(w)}(x-w) \approx \frac{1}{1 - \frac{f''(w)}{2f'(w)}(x-w)}$$

Putting this together gives

$$\frac{1}{f'(w)}\frac{f(x)-f(w)}{x-w} \approx \frac{1}{1 - \frac{f''(w)}{2f'(w)}(x-w)}$$

which becomes

$$f(x) \approx f(w) + \frac{f'(w)(x-w)}{1 - \frac{f''(w)}{2f'(w)}(x-w)}$$

which is expressible as $\frac{ax+b}{cx+d}$. Notice if $f'(w) = 0$, this becomes $f(x) \approx f(w)$, and if $f''(w) = 0$ it becomes the linear approximation.

Another solution method is to rewrite the Möbius transformation $g(x)$ to only depend on three parameters, write the system of equations

$$g(w) = f(w), \quad g'(w) = f'(w), \quad g''(w) = f''(w),$$

and then solve for the three parameters.