

Problem of the week #8: Solution

While a point is rotated around an axis, it traces out a circle which is contained in a plane perpendicular to the axis. In particular, its velocity vector must be orthogonal to the axis at all times.

The key idea is that for two balls to be rotating against each other without slipping, the velocity vector at their point of contact must be the same with respect to both rotations. Thus the axes must both be contained in the plane perpendicular to this velocity vector. Since the angles the axes make with the line segment joining the balls' centers are acute, the axes cannot be parallel, so they must intersect at a point.

To find the speed of a particular point under a rotation, we may find the circumference of the circle it traverses and multiply by the angular velocity ω in revolutions per unit time. (This is the usual distance-rate-time formula, where distance = revolutions \times circumference.)

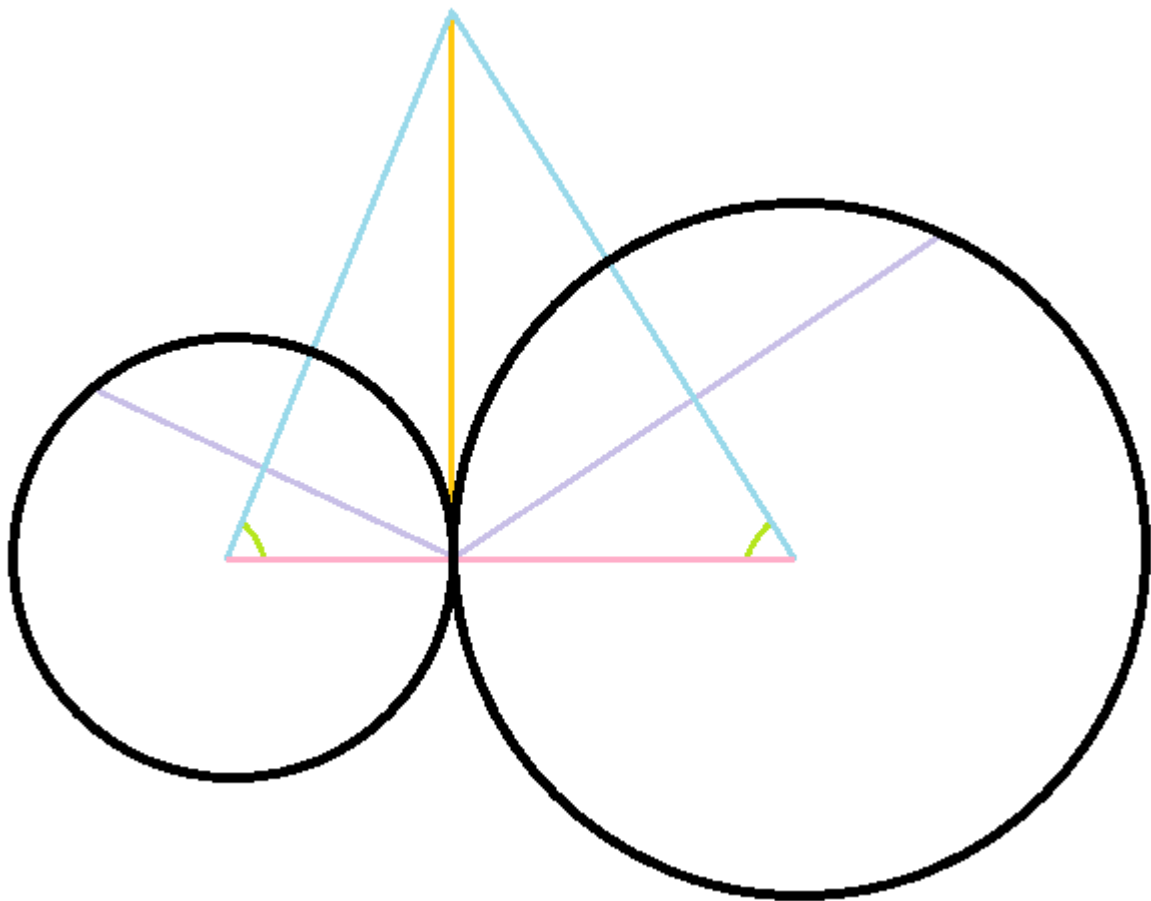
Suppose balls have radii, acute angles (between axes of rotation and the line segment joining their centers) and angular velocities R_1, θ_1, ω_1 and R_2, θ_2, ω_2 respectively. Let r_1 and r_2 be the smaller radii of the circles which are traced out by the point of contact under rotation.

By drawing a triangle between a ball's center, the smaller circle's center, and the point of contact between the balls, we see the smaller radius is $r_i = R_i \sin \theta_i$ for $i = 1, 2$. Setting the speeds v_1 and v_2 equal to each other then gives the equation $2\pi\omega_1 R_1 \sin \theta_1 = 2\pi\omega_2 R_2 \sin \theta_2$.

Then, if the axes' intersection is in the plane between the balls, the line segment joining it to the point of contact must be perpendicular to the other line segment joining the balls' centers. Thus, it is an altitude of the triangle formed by the axes and the line segment joining the centers, which by trigonometry is $R_1 \tan \theta_1 = R_2 \tan \theta_2$.

Dividing the last two equations yields $\omega_1 \cos \theta_1 = \omega_2 \cos \theta_2$.

Cross section in plane perpendicular to velocity vector:



Pink: spherical radii R_1 and R_2

Green: axis angles θ_1 and θ_2

Blue: axes of rotation

Violet: circular radii r_1 and r_2

Orange: altitude of large triangle