Problem of the week #7: Solutions

Solution 1. Define $f(T) = T^3 + aT^2 + bT + c$. By the fundamental theorem of algebra, it can be factored as $f(T) = (T-\alpha)(T-\beta)(T-\gamma)$ for three (not necessarily distinct) roots α, β, γ. Expanding yields:

$$f(T) = T^3 - (\alpha+\beta+\gamma)T^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)T - (\alpha\beta\gamma).$$

Vieta's formulas state that, for monic (i.e. leading coefficient 1) polynomials of *any* degree, each coefficient is equal to $\pm$ a corresponding **elementary symmetric polynomial** of the roots α, β, γ.

In this case, we have:

$$\begin{aligned} \alpha+\beta+\gamma &= -a \\ \alpha\beta+\beta\gamma+\gamma\alpha &= b \\ \alpha\beta\gamma &= -c \end{aligned}$$

On the other hand, define $g(T) = (T-\alpha^2)(T-\beta^2)(T-\gamma^2)$, and assume it expands as $g(T) = T^3 + AT^2 + BT + C$, then Vieta's formulas say

$$\begin{aligned} \alpha^2+\beta^2+\gamma^2 &= -A \\ (\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2 &= B \\ (\alpha\beta\gamma)^2 &= -C \end{aligned}$$

The easiest to find is $C = -(\alpha\beta\gamma)^2 = -c^2$.

Next, notice $(-a)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha)$ (after regrouping and combining like terms) which is $-A+2b$, and so $A = 2b-a^2$.

Finally, $b^2 = (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 + 2(\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2)$, by the same token. The latter part may be factored as $2\alpha\beta\gamma(\alpha+\beta+\gamma)$, so this equation states $b^2 = B + 2(-c)(-a)$, and thus $B = b^2 - 2ac$.

Putting it all together, we conclude

$$g(T) = T^3 + (2b-a^2)T^2 + (b^2-2ac)T - c^2.$$

Solution 2. The formula $A^2 - B^2 = (A - B)(A + B)$, which says a difference of squares factors as a product of conjugates, may be used:

$$g(T) = (T - \alpha^2)(T - \beta^2)(T - \gamma^2)$$

$$= (\sqrt{T} - \alpha)(\sqrt{T} + \alpha) \cdot (\sqrt{T} - \beta)(\sqrt{T} + \beta) \cdot (\sqrt{T} - \gamma)(\sqrt{T} + \gamma)$$
$$= (\sqrt{T} - \alpha)(\sqrt{T} - \beta)(\sqrt{T} - \gamma) \cdot (\sqrt{T} + \alpha)(\sqrt{T} + \beta)(\sqrt{T} + \gamma),$$

valid for $T \geq 0$, or even for $T < 0$ if we adopt the convention $\sqrt{-x} = ix$ whenever $-x$ is negative. The first three factors are $f(\sqrt{T})$, however the last three terms have $+$ signs. To remedy this, multiply by $(-1)^4$ and distribute the (-1)s out like so:

$$(\sqrt{T} + \alpha)(\sqrt{T} + \beta)(\sqrt{T} + \gamma) = -(-\sqrt{T} - \alpha)(-\sqrt{T} - \beta)(-\sqrt{T} - \gamma).$$

Thus, we have $g(T) = -f(\sqrt{T})f(-\sqrt{T})$. Multiplying this out,

$$g(T) = (T^{3/2} + aT + bT^{1/2} + c)(T^{3/2} - aT + bT^{1/2} - c)$$

$$= T^3 + (2b - a^2)T^2 + (b^2 - 2ac)T - c^2$$

The fractional powers cancel out in the end. (Interpret $T^{3/2}$ and $T^{1/2}$ as placeholders for $T\sqrt{T}$ and $\sqrt{T}$ for negative numbers if necessary.)