Problem of the week #6: Solutions

The divergent sums we want meaningful values for are

$$\begin{array}{ccccccccccc} A & = & 1 & + & 2 & + & 3 & + & 4 & + & \cdots \\ B & = & 1 & + & 2^2 & + & 3^2 & + & 4^2 & + & \cdots \\ C & = & 1 & + & 2^3 & + & 3^3 & + & 4^3 & + & \cdots \end{array}$$

Solution 1. Differentiating the geometric series

$$\frac{x}{1+x} = x - x^2 + x^3 - x^4 + \cdots$$

and then multiplying by x gives

$$\frac{x}{(1+x)^2} = x - 2x^2 + 3x^3 - 4x^4 + \cdots$$

Doing this twice more yields two more series expansions

$$\frac{x(1-x)}{(1+x)^3} = x - 2^2x^2 + 3^2x^3 - 4^2x^4 + \cdots$$

$$\frac{1-4x+x^2}{(1+x)^4} = x - 2^3x^2 + 3^3x^3 - 4^3x^4 + \cdots$$

Setting $x = 1$ in the last three equations yields

$$\frac{1}{4} = 1 - 2 + 3 - 4 + \cdots$$

$$0 = 1 - 2^2 + 3^2 - 4^2 + \cdots$$

$$-\frac{1}{8} = 1 - 2^3 + 3^3 - 4^3 + \cdots$$

To obtain the regularized values of the non-alternating versions of these sums, we may use zero-padding and linearity. In particular,

$$a_1 + a_2 + a_3 + a_4 + \cdots = 0 + a_1 + 0 + a_2 + 0 + a_3 + 0 + \cdots$$

Then we may rewrite

$$\begin{aligned}
\frac{1}{4} &= 1-2+3-4+\cdots \\
&= \begin{array}{r}(1+2+3+4+\cdots) \\ -2(0+2+0+4+\cdots)\end{array} \\
&= \begin{array}{r}(1+2+3+4+\cdots) \\ -4(0+1+0+2+\cdots)\end{array} \\
&= A-4A \quad = \quad -3A
\end{aligned}$$

which implies $A=-\frac{1}{12}$ and then

$$\begin{aligned}
0 &= 1-2^2+3^2-4^2+\cdots \\
&= \begin{array}{r}(1+2^2+3^2+4^2+\cdots) \\ -2(0+2^2+0+4^2+\cdots)\end{array} \\
&= \begin{array}{r}(1+2^2+3^2+4^2+\cdots) \\ -8(0+1^2+0+2^2+\cdots)\end{array} \\
&= B-8B \quad = \quad -7B
\end{aligned}$$

which implies $B=0$ and then

$$\begin{aligned}
-\frac{1}{8} &= 1-2^3+3^3-4^3+\cdots \\
&= \begin{array}{r}(1+2^3+3^3+4^3+\cdots) \\ -2(0+2^3+0+4^3+\cdots)\end{array} \\
&= \begin{array}{r}(1+2^3+3^3+4^3+\cdots) \\ -16(0+1^3+0+2^3+\cdots)\end{array} \\
&= C-16C \quad = \quad -15C
\end{aligned}$$

which implies $C=\frac{1}{120}$.

Solution 2. We may find A, B, C and their alternating versions, which we'll call X, Y, Z, without differentiating the geometric series formula.

$$\begin{aligned} W &= 1 - 1 + 1 - 1 + \cdots \\ X &= 1 - 2 + 3 - 4 + \cdots \\ Y &= 1 - 2^2 + 3^2 - 4^2 + \cdots \\ Z &= 1 - 2^3 + 3^3 - 4^3 + \cdots \end{aligned}$$

The geometric series formula already gives $W = \frac{1}{2}$, but also

$$2W = \begin{array}{c} (1 - 1 + 1 - 1 + \dots) \\ +\,(0 + 1 - 1 + 1 - \cdots) \end{array} = 1$$

which implies $W = \frac{1}{2}$ as well. Similarly,

$$2X = \begin{array}{c} (1 - 2 + 3 - 4 + \dots) \\ +\,(0 + 1 - 2 + 3 - \cdots) \end{array} = W$$

and $2X = \frac{1}{2}$ implies $X = \frac{1}{4}$.

The next one requires splitting up into two previous alternating sums:

$$\begin{aligned} 2Y &= \begin{array}{c} (1 - 2^2 + 3^2 - 4^2 + \dots) \\ +\,(0 + 1^2 - 2^2 + 3^2 - \cdots) \end{array} \\ &= 1 - 3 + 5 - 7 + \cdots \\ &= \begin{array}{c} (1 - 1 + 1 - 1 + \dots) \\ -2(0 + 1 - 2 + 3 - \cdots) \end{array} \\ &= W - 2X \end{aligned}$$

and $2Y = \frac{1}{2} - 2(\frac{1}{4})$ implies $Y = 0$.

And the last one requires splitting into three previous sums.

$$
\begin{aligned}
2Z &= \quad (1 - 2^3 + 3^3 - 4^3 + 5^3 - \dots) \\
&\quad + (0 + 1^3 - 2^3 + 3^3 - 4^3 + \cdots) \\
&= \quad 1 - 7 + 19 - 37 + 61 - \cdots \\
&= \quad (1 - 1 + 1 - 1 + 1 - \dots) \\
&\quad -6(0 + 1 - 3 + 6 - 10 + \cdots)
\end{aligned}
$$

Notice 1, $3 = 1+2$, $6 = 1+2+3$, $10 = 1+2+3+4$ are the triangular numbers, which satisfy $1+2+3+\cdots+n = \frac{1}{2}(n^2+n)$. Continuing,

$$
\begin{aligned}
&= \quad (1 - 1 + 1 - 1 + 1 - \dots) \\
&\quad -3(0 + 1 - 2 + 3 - 4 + \cdots) \\
&\quad -3(0 + 1 - 4 + 9 - 16 + \cdots) \\
&= \quad W - 3X - 3Y
\end{aligned}
$$

and $2Z = \frac{1}{2} - 3(\frac{1}{4}) - 3(0)$ implies $Z = -\frac{1}{8}$.

We could have also said $(n+1)^3 - n^3 = 3n^2 + 3n + 1$ to similar effect.

Then A, B, C can be gotten from X, Y, Z as in Solution 1.

The Riemann zeta function, defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\mathrm{Re}(s) > 1$ (its abscissa of convergence), also exists for complex numbers s with real parts less than or equal to 1, except at $s = 1$ itself. This is like the situation for $(1-x)^{-1} = \sum_{n=1}^{\infty} x^n$, where the series converges for $|x| < 1$ but the function exists for all x except $x = 1$. The process of extending the domain of a function in the complex plane is called **analytic continuation**, common in regularization.

In general, $\zeta(-n) = 1 + 2^n + 3^n + 4^n$ have the formula $\zeta(-n) = B_{n+1}/(n+1)$, where the **Bernoulli numbers** B_n appear in the exponential generating function $x/(e^x - 1) = \sum_{n=0}^{\infty}(B_n/n!)x^n$, as well as coefficients in all of the so-called Faulhaber polynomials $P_s(n)$ defined by $P_s(n) = \sum_{k=1}^{n} k^s = 1 + 2^s + 3^s + \cdots + n^s$, e.g. $P_1(n) = \frac{1}{2}(n^2 + n)$.