Problem of the week #6: Background

Addition is not a priori defined for infinitely many summands, it is a posteriori defined as a limit of partial sums. On this interpretation, there are series which don't have values (the limits don't exist) which are said to diverge. There many kinds of divergence or even strange kinds of convergence. Consider the following examples:

- $1+2+3+4+\cdots$: diverges to $+\infty$, as do the terms
- $1-2+3-4+\cdots$: diverges, but not to $+\infty$. or $-\infty$; the terms and partial sums are unbounded but oscillate sign.
- $1+1+1+1+\cdots$: diverges to $+\infty$, but terms are bounded.
- $1-1+1-1+\cdots$: diverges, but bounded terms and partial sums, and if regrouped to $(1-1)+(1-1)+\cdots$ converges.
- $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges to $+\infty$, but ever so slowly: terms tend to 0, nth partial sum approximately $\ln n$
- $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ converges conditionally to $\ln 2$; permuting infinitely many terms can cause it to converge to potentially *any* other real number (Riemann rearrangement theorem).
- $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$ converges to $\frac{\pi^2}{6}$, first noticed and explained by Euler in the so-called Basel problem. It is an example of a p-series and a particular value of the Riemann zeta function $\zeta(s)$.

If $\mathcal{S}$ denotes the set of all infinite sequences of real numbers, and $\mathcal{C}$ the set of all convergent sequences, then $\mathcal{S}$ is a real vector space and $\mathcal{C}$ is a subspace of $\mathcal{S}$. The summation operator Σ which outputs the limit of partial sums of a sequence is a linear map $\mathcal{C} \to \mathbb{R}$.

An extension of Σ to a linear map $\mathcal{V} \to \mathbb{R}$ on a larger subspace $\mathcal{V}$ of $\mathcal{S}$ is called a **summability method**. This allows us to assign finite values to divergent series. The axiom of choice implies summability methods exist, but this doesn't inherently guarantee they're anything but arbitrary. Fortunately, there are ones of interest.

Summability methods are a subset of a broader class of methods called **regularization**, which resolve infinities out of calculations - not just sums, but also products, or functional integrals (called Feynman path integrals in quantum theory) - to get meaningful results. They can even bridge the gap between theoretical calculations and physical observations, as in the Casimir effect.

Baby-zeta regularization will be our name for the summability method to be described momentarily. Grown-up zeta regularization requires exploring Dirichlet series, which we will (tragically) substitute with Taylor series for simplicity. Whenever zeta and baby-zeta regularization both work, they give the same result, however baby-zeta won't work as often, and grown-up zeta won't behave as nicely.

Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series for which

$$f(x) = \sum_{n=1}^{\infty} a_n x^n$$

is true within a positive radius of convergence for some analytic (i.e. nice) function $f(x)$. Then $f(1)$, if it exists, is called the baby-zeta regularized value of the series. For example, take the geometric series

$$\frac{x}{1+x} = x - x^2 + x^3 - x^4 + \cdots$$

While this power series only converges for $|x| < 1$, the function on the left is defined for all values except $x = -1$, so setting $x = 1$ gives

$$\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots$$

Heuristically, if we split it into $(1+0+1+0+\cdots)-(0+1+0+1+\cdots)$, then delete the 0s to get $(1+1+1+1+\cdots)-(1+1+1+1+\cdots)$ we should expect 0, which is a contradiction. This is resolved two ways:

- Grown-up zeta does assign $1 + 1 + \cdots$ the value $-\frac{1}{2}$, but does not allow infinitely many 0s to be introduced or discarded.
- Baby zeta allows introducing/discarding 0s arbitrarily, but does not assign $1 + 1 + \cdots$ any value so there is no contradiction.

This unlocks new tools to define the zeta regularization of $\sum_{n=1}^{\infty} a_n$ besides simply analyzing the function $f(x) = \sum_{n=1}^{\infty} a_n x^n$. Both grown-up and baby zeta regularization are linear:

$$\sum_{n=1}^{\infty}(a_n + b_n) = \left(\sum_{n=1}^{\infty} a_n\right) + \left(\sum_{n=1}^{\infty} b_n\right), \quad \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n.$$

And introducing 0s between terms does not affect a baby zeta value, even if it is infinitely many 0s in total. Ultimately, this means even if $f(x) = \sum_{n=1}^{\infty} a_n x^n$ does not have $f(1)$ defined, $\sum_{n=1}^{\infty} a_n$ may still potentially possess a zeta regularized value after all.

Integrating or differentiating the aforementioned geometric series gives

$$\ln\left(\frac{1}{1+x}\right) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\frac{x}{(1+x)^2} = x - 2x^2 + 3x^3 - 4x^4 + \cdots$$

These are altered versions of the Mercator series and Newton-binomial series, respectively. Plugging $x = 1$ into the first gives the aforementioned series for $\ln 2$, although that one is already convergent.