Problem of the week #5: Solution

(a) and (b):

First, let's find how many regular hexagons are incident to $(1,0,0,0)$, the rightmost point in the five w slices (see background document).

The distances from $(1,0,0,0)$ to other points are as follows:

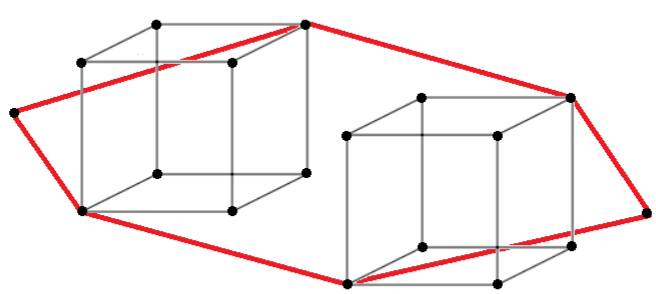
- $\sqrt{1}$ for those in the $w = +\frac{1}{2}$ slice,
- $\sqrt{2}$ for those in the $w = 0$ slice,
- $\sqrt{3}$ for those in the $w = -\frac{1}{2}$ slice, and
- $\sqrt{4}$ for those in the $w = -1$ slice.

The second point in a regular hexagon must be 1 unit away, which forces it to be any of the eight points in the $w = \frac{1}{2}$ slice.

The third point must be 1 unit away from the second and $\sqrt{3}$ units from the first, forcing it to be the cube vertex in the $w = -\frac{1}{2}$ slice corresponding to the one chosen second in the $w = \frac{1}{2}$ slice. So if we picked $\frac{1}{2}(1,1,1,1)$ for the second point, then the third must be $\frac{1}{2}(-1,1,1,1)$.

For the next three points, use the fact the regular hexagon is symmetric across the origin. This means it must contain the opposites of the three points we already picked. This leaves us with one hexagon:

$$H = \{(\pm 1, 0, 0, 0), \tfrac{1}{2}(\pm 1, \underbrace{\pm 1, \pm 1, \pm 1}_{\text{same sign}})\}.$$



Notice the entire hexagon was determined by the choice of second point. There was some redundancy in this choice, however, since the hexagon ultimately ended up having a pair of vertices antipodal within the $w = \frac{1}{2}$ slice's cube, $\frac{1}{2}(1, 1, 1, 1)$ and $\frac{1}{2}(1, -1, -1, -1)$.

Therefore, the number of hexagons incident to $(1, 0, 0, 0)$ is the number of antipodal pairs of vertices of the cube, which is 4. By symmetry, every point of the 24-cell is incident to 4 hexagons. Therefore,

$$(\#\text{hexagons})(\#\text{points per hexagon}) = (\#\text{points})(\#\text{hexagons per point})$$

implies the number of hexagons is $24 \times 4 / 6 = 16$. Alternatively, we can note that the second choice of vertex determining the rest of the hexagon is equivalent to a saying single edge (between the first and second vertex) fully determining the hexagon. Therefore,

$$(\#\text{hexagons})(\#\text{edges per hexagon}) = \#\text{edges}$$

implies the number of hexagons is $96 / 6 = 16$. The edge count

$$2(8 + 12 + 24) + 8 = 96$$

follows from counting edges between the five w slices:

- 8 edges from $w = 1$ to the $w = \frac{1}{2}$,
- 12 edges within the $w = \frac{1}{2}$ slice,
- 24 edges between the $w = \frac{1}{2}$ slice and the $w = 0$ slice:
  - 8×3 by picking a vertex in $w = \frac{1}{2}$ then $w = 0$, or
  - 6×4 by picking a vertex in $w = 0$ then $w = \frac{1}{2}$,
- by symmetry, ditto for the negative side of $w = 0$,
- 8 edges between the $w = \pm\frac{1}{2}$ slices.

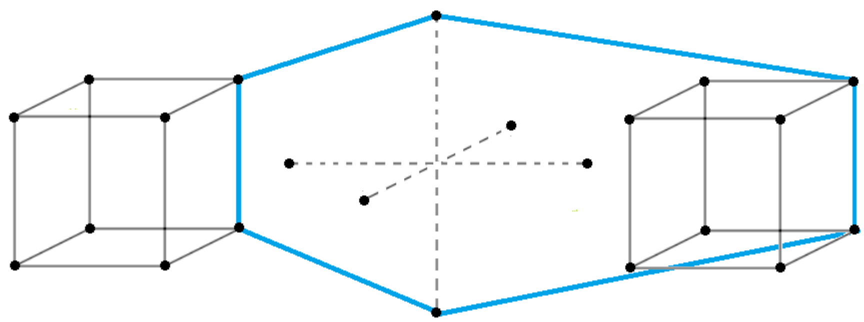
See background document; there it also states the edge count is 96.

In summary, every edge is contained in exactly one hexagon, so there are 16 hexagons and they form a partition of the 96 edges.

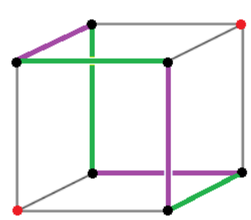
(c) and (d):

We will construct a bundle by first picking a hexagon H_1, then a disjoint hexagon H_2, and so on. At each stage, we note how many hexagons we have to choose from. By symmetry, we expect the number of hexagons available at any stage is independent of which hexagons were chosen before. By the fundamental counting principle, we multiply the numbers to count the ordered bundles (H_1, H_2, H_3, H_4).

Even before this, though, there are 16 hexagons total and 4 through $w = \pm 1$; where are the other 12? Pick any of the 12 edges in the $w = \frac{1}{2}$ slice, pick the corresponding edge in the $w = -\frac{1}{2}$ slice, then connect them to the endpoints of the parallel axis in the $w = 0$ slice:



We've already picked H_1 out of 16 options. Our next hexagon H_2 cannot use either of the vertices in $w = \pm 1$ so must entirely be contained in the middle three slices. It must also avoid the cube-antipodal points already used in the $w = \pm\frac{1}{2}$ slices, as well as all of the edges connected to them. This leaves the following six edges available:



Any of the six edges may be chosen, but the next three hexagons must be vertex-disjoint, so we must choose all three edges of one or the other color and determine three hexagons H_2, H_3, H_4 from them.

Picking the three green edges, we have the following bundle:

$$\begin{aligned}
H_1 &= \{(\pm1,0,0,0), \tfrac{1}{2}(\pm1,\overbrace{\pm1,\pm1,\pm1}^{\text{same sign}})\},\\
H_2 &= \{(0,\pm1,0,0), \tfrac{1}{2}(\pm1,\pm1,-1,+1)\},\\
H_3 &= \{(0,0,\pm1,0), \tfrac{1}{2}(\pm1,+1,\pm1,-1)\},\\
H_4 &= \{(0,0,0,\pm1), \tfrac{1}{2}(\pm1,-1,+1,\pm1)\}.
\end{aligned}$$

There are 4! permutations of (H_1, H_2, H_3, H_4), and when constructing the bundle we had 16 options for H_1 followed by $3! = 6$ permutations of 2 color choices for the remaining three hexagons H_2, H_3, H_4, thus

$$\#\text{bundles} = \frac{16 \times 6 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = 8.$$

More Background: Group Theory

Viewing (w, x, y, z) as a quaternion $w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the 24-cell Q_{24} is a finite group under multiplication which contains the usual quaternion group $Q_8 = \{\pm, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. The elements of orders $1, 2, 3, 4, 6$ correspond to vertices a distance of $\sqrt{0}, \sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}$ away from 1 in the slices $w = 1, -1, -\frac{1}{2}, 0, \frac{1}{2}$ respectively.

The order six subgroups of Q_{24}, which are cyclic, are the four hexagons through ±1, and are all conjugate. The 16 hexagons are the cosets of these subgroups, and the 8 bundles are the left and right coset spaces of these hexagon subgroups. (Note every left coset is a right coset of a conjugate subgroup, but left coset spaces are not right coset spaces.)