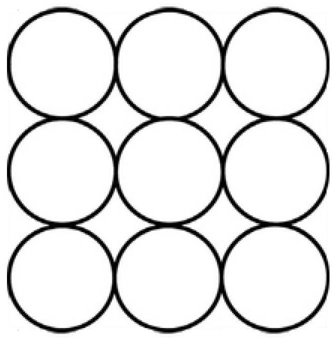
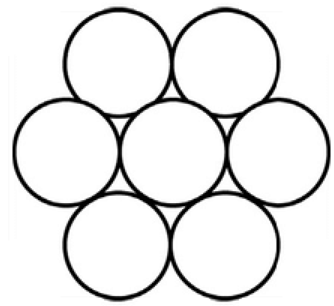
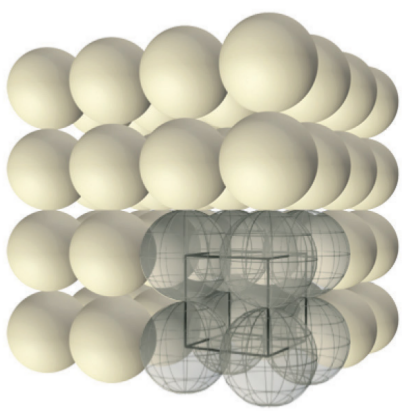
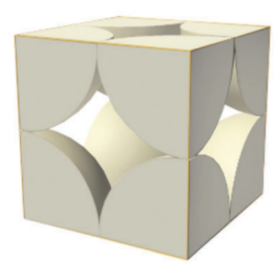
Problem of the week #5: Background

Points with integer coordinates form the grid $\mathbb{Z}^2$ within the plane $\mathbb{R}^2$.

Placing a circle of diameter 1 centered at each grid point results in a square packing. The most efficient packing, however, is hexagonal.

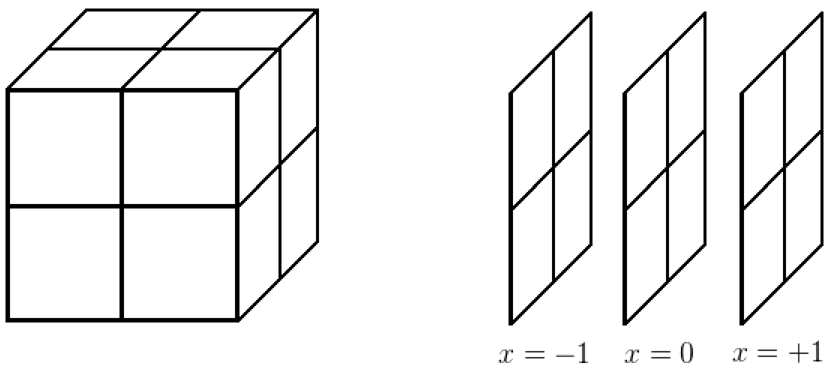
Tangentially, the so-called kissing number problem asks how many circles can lie around one of the same size. The answer is that the kissing number is 6, arising from a hexagonal arrangement of adjacent-tangent circles. The problem generalizes to higher dimensions.

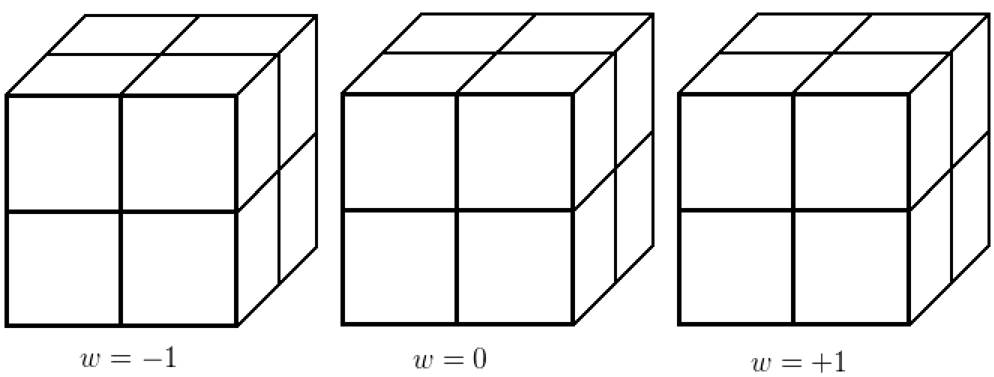
Similarly, a cubic arrangement of spheres is inefficient. Six spheres lie tangent to any given one, but there is extra space between them.

Taking eight sphere centers to be the vertices of a cube, there is not enough room to fit an extra sphere in the middle: the diagonal through a cube is $\sqrt{3}$ which cannot fit two radii and one diameter (i.e. $\sqrt{3} < 2$).

To understand this arrangement in four dimensions, first notice how in three dimensions a $2 \times 2 \times 2$ cube centered at $(0, 0, 0)$ has three 2×2 square cross-sections corresponding to x coordinates $-1, 0, +1$; between each adjacent pair of 1×1 squares there is a $1 \times 1 \times 1$ cube.



Generalizing to four dimensions, points are represented by (w, x, y, z), and a $2 \times 2 \times 2 \times 2$ tesseract has three $2 \times 2 \times 2$ cube cross-sections corresponding to w coordinates $-1, 0, 1$; between each adjacent pair of $1 \times 1 \times 1$ cubes there is a $1 \times 1 \times 1 \times 1$ tesseract.



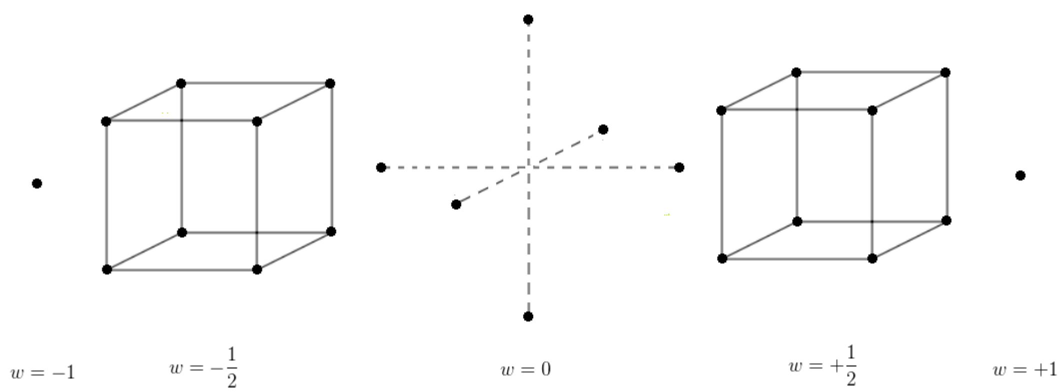
The diagonal between opposite vertices in a unit tesseract is $\sqrt{4} = 2$, since the distance formula in two and three dimensions generalizes to four dimensions. This is just enough for two radii and a diameter if we pack hyperspheres with diameter 1 at all the points with integer coordinates (the grid $\mathbb{Z}^4$ within $\mathbb{R}^4$), therefore we can snugly fit a hypersphere within all sixteen unit tesseracts.

These sixteen hyperspheres centered at $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$, plus the eight hyperspheres centered at the permutations of $(\pm 1, 0, 0, 0)$, are adjacent-tangent like the six circles of the 2D hexagonal packing.

Just as $\mathbb{R}^2$ has polygons and $\mathbb{R}^3$ has polyhedra, $\mathbb{R}^4$ has polychora, and in general, $\mathbb{R}^n$ has polytopes. The centers of these 24 hyperspheres form the vertices of a polychoron called the 24-cell.

Edges are between centers of tangent hyperspheres, faces are between coplanar centers of mutually tangent hyperspheres, and so on. Two hyperspheres are tangent precisely when their centers are 1 unit apart.

To understand their arrangement, consider the five 3D cross-sections containing the hypersphere centers with w coordinates $-1, -\frac{1}{2}, 0, +\frac{1}{2}, +1$:



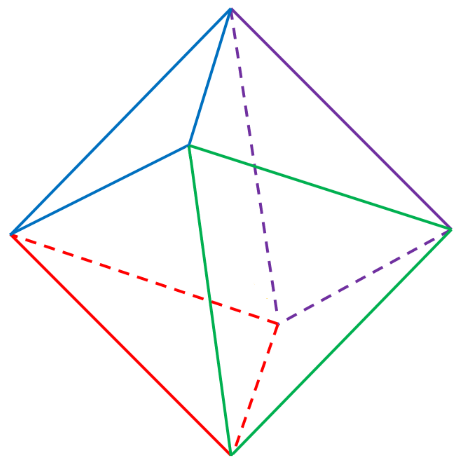
The distance between any two points is one of $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}$.

The six vertices in the $w = 0$ slice are not connected to each other, but are connected to the $w = \pm\frac{1}{2}$ slices on either side - the top vertex is connected to the top four vertices on each cube, the front vertex is connected to the front four vertices on each cube, and so on.

Each edge of the two cubes is also an edge of the 24-cell, each vertex of a cube is connected to the corresponding vertex of the other cube, and the two isolated vertices in the $w = \pm 1$ slices are connected to every vertex in their neighboring cube.

The square faces of the cube are not faces of the 24-cell. Any three edges connecting three vertices bound a triangular face, for example any pair of adjacent vertices in a cube ($w = \pm\frac{1}{2}$) and the neighboring isolated vertex ($w = \pm 1$) form a face.

While the cubes' square faces are not faces of the 24-cell, their vertices are connected to one vertex in a neighboring slice on either side forming an octahedral cell, as pictured below. In the 24-cell, all faces are triangular and all cells are octahedral.



The 24-cell is perfectly symmetrical: any of its vertices, edges, faces or cells correspond to any other with respect to a rotational symmetry (in physics this is known as isotropy). So, for instance, since the leftmost vertex in the $w = -1$ slice is adjacent to 8 vertices (which are all in the $w = -\frac{1}{2}$ slice), every vertex is connected to 8 others.

Or, since the leftmost vertex is incident to 6 octahedra (one for each square face in the $w = -\frac{1}{2}$ slice), the same is true of every vertex, and since there are 24 vertices and 6 per octahedron, there are $6 \cdot 24/6 = 24$ octahedral cells. This logic lets us fill in the configuration matrix:

	V	E	F	C
V	24	2	3	6
E	8	96	3	12
F	12	3	96	8
C	6	3	2	24

The diagonal entries say how many vertices, edges, faces and cells there are. The off-diagonal entries describe incidence, for instance the V row and C column says there are 6 vertices incident to every cell.

Exercise. Verify the configuration matrix above. Remember to use symmetry! Extra: compute the matrix for the five Platonic solids.