Problem of the week #4: Solution

The number of trailing zeros in a number m, represented in binary, equals the number of times it is divisible by 2, or equivalently the power of 2 in its prime factorization.

Let $v_2(m)$ be the power of 2 in m's prime factorization. In number theory this is called the *2-adic valuation*. Much like a logarithm, it satisfies the product rule $v_2(ab) = v_2(a) + v_2(b)$. Therefore the valuation of $m = 1^1 2^2 3^3 \cdots 2049^{2049}$ is equal to the sum

$$1v_2(1) + 2v_2(2) + 3v_2(3) + \cdots + 2049v_2(2049)$$

We may tally the valuations $v_2(k)$ for $k = 1, \cdots, 16$ as in the below table on the left. To multiply k times $v_2(k)$ we may replace each dot with a k and insert plus signs, as on the right:

1	○	○	○	○		
2	●	○	○	○		2
3	○	○	○	○		
4	●	●	○	○		4 + 4
5	○	○	○	○		
6	●	○	○	○		6
7	○	○	○	○		
8	●	●	●	○	$\Longrightarrow$	8 + 8 + 8
9	○	○	○	○		
10	●	○	○	○		10
11	○	○	○	○		
12	●	●	○	○		12 + 12
13	○	○	○	○		
14	●	○	○	○		14
15	○	○	○	○		
16	●	●	●	●		16 + 16 + 16 + 16

Evaluating $n = v_2(m)$, then, amounts to adding up all the numbers scattered above on the right. Instead of grouping the terms in rows, giving $2 + 4 \cdot 2 + 6 + 8 \cdot 3 + 10 + 12 \cdot 2 + 14 + 16 \cdot 4 + \cdots$, we will group the terms in columns because the column sums have a formula.

Grouping the summands according to columns on the last page,

$$
\begin{array}{lr}
 & (2+4+6\ +\ 8\ +\cdots+2048) \\
 & (4+8+12+\cdots+2048) \\
 & (8+16+\cdots+2048) \\
 & (16+\cdots+2048) \\
 & \vdots \\
+ & (2048) \\
\hline
= & n
\end{array}
$$

From here we may factor out common factors:

$$
\begin{array}{lr}
 & 2(1+2+3+\cdots+1024) \\
 & 4(1+2+3+\cdots+\ \ 512) \\
 & 8(1+2+3+\cdots+\ \ 256) \\
 & 16(1+2+3+\cdots+\ \ 128) \\
 & \vdots \\
+ & 2048(1) \\
\hline
= & n
\end{array}
$$

Note 2048 is a power of 2, by hand calculation:

e	2^e
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024
11	2048

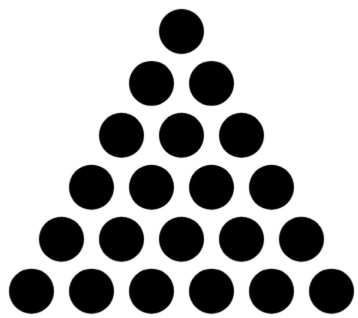
And so it is revealed that 2049 is more than a *Blade Runner* reference; it is closer to a perfect power of 2 than 2019 happens to be.

At this point we need to use the following:

Lemma. The nth *triangular number* is given by the formula

$$T_n := 1 + 2 + 3 + \cdots + n \quad = \frac{n(n+1)}{2}.$$

They count how many balls are in a triangular stack (by row):



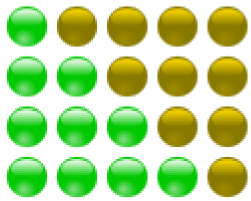
(*Proof 1.*) T_n counts how many subsets $\{1, 2, 3, \cdots, n+1\}$ has of the form $\{a, b\}$. If we pick a first then b, we have $n+1$ choices for a then n remaining choices for b, but then we must divide by 2 to undo our overcounting since $\{a, b\} = \{b, a\}$; this gives $n(n+1)/2$. Or, make the following arrangement and count by rows:

$$\begin{array}{c}
\{1,2\} \\
\{1,3\}, \{2,3\} \\
\{1,4\}, \{2,4\}, \{3,4\} \\
\{1,5\}, \{2,5\}, \{3,5\}, \{4,5\} \\
\{1,6\}, \{2,6\}, \{3,6\}, \{4,6\}, \{5,6\} \\
\{1,7\}, \{2,7\}, \{3,7\}, \{4,7\}, \{5,7\}, \{6,7\} \\
\vdots
\end{array}$$

(*Proof 2.*) There is an oft-told story which says that when a young Carl Friedrich Gauss (considered one of the greatest mathematicians of all time) was a schoolboy, his teacher gave the students busywork by asking them to add the numbers 1 through 100, which Gauss solved immediately with the trick of adding the sum to itself in reverse order.

$$\begin{array}{rcccccccccc} S & = & 1 & + & 2 & + & \cdots & + & 99 & + & 100 \\ +S & = & 100 & + & 99 & + & \cdots & + & 2 & + & 1 \\ \hline 2S & = & 101 & + & 101 & + & \cdots & + & 101 & + & 101 \end{array}$$

Summing gives $2S = 100(101)$. This generalizes to $2T_n = n(n+1)$, and may be visualized by combining two triangular stacks:



$$T_4 + T_4 = 4 \times 5$$

Applying the lemma to our aforementioned column sum for n,

$$\begin{aligned} n &= 2\left(\tfrac{1024\cdot 1025}{2}\right) + 4\left(\tfrac{512\cdot 513}{2}\right) + 8\left(\tfrac{256\cdot 257}{2}\right) + \cdots + 2048\left(\tfrac{1\cdot 2}{2}\right) \\ &= 1024(1025 + 513 + 257 + \cdots + 2) \end{aligned}$$

For this we may employ yet another formula,

Lemma. The *geometric sum formula* for the kth partial sum of a geometric sequence with first term 1 and common ratio r is:

$$S = 1 + r + r^2 + \cdots + r^{k-1} = \frac{r^k - 1}{r - 1}.$$

(*Proof.*) Compare S with its multiple rS:

$$\begin{array}{rcccccccccc} S & = & 1 & + & r & + & r^2 & + & \cdots & +r^{k-1} & \\ rS & = & & & r & + & r^2 & + & \cdots & +r^{k-1} & + \; r^k \end{array}$$

Subtracting gives $rS - S = r^k - 1$. Applying with $r = 2$ and $k = 11$,

$$\begin{aligned} n &= 2^{10}\big((2^{10}+1) + (2^9+1) + \cdots + (2^0+1)\big) \\ &= 2^{10}\big((2^{10} + 2^9 + \cdots + 2^0) + (1 + 1 + \cdots + 1)\big) \\ &= 2^{10}\left(\frac{2^{11}-1}{2-1} + 11\right) = 2^{10}(2^{11} + 2^3 + 2) = 2^{21} + 2^{13} + 2^{11} \end{aligned}$$

expressed in binary is 1000000010100000000000_2.