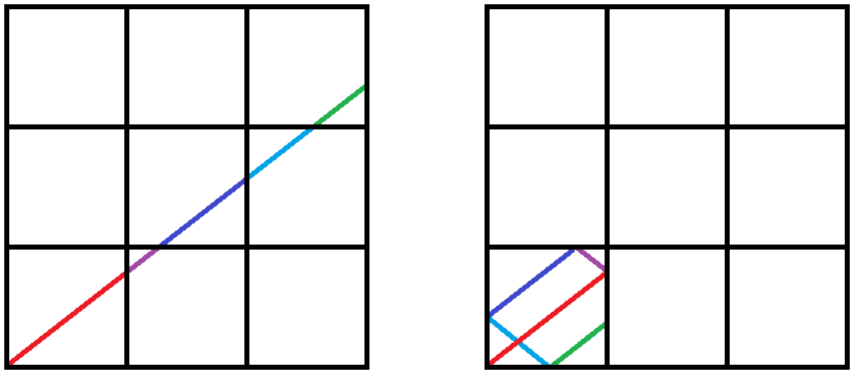
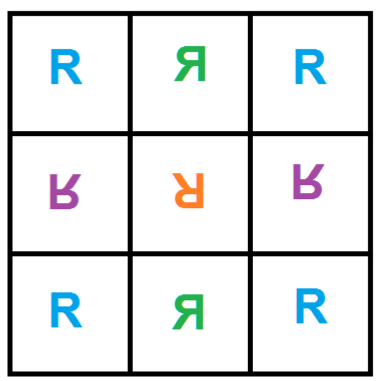
Problem of the week #3: Solution

Compare the trajectory of a point particle bouncing around the unit square with that of a point particle free to traverse the plane:



As the particle crosses the first edge, its path (purple) may be flipped across the edge to get the would-be trajectory had the particle instead bounced. Generalizing, the points in any square correspond to points in an adjacent square by flipping across the common edge. To illustrate, consider the effect of repeatedly flipping the letter ‘R’ as follows:



The pattern is seen to repeat every 2 units. Therefore, after computing where the point would be with no bouncing, we may subtract even numbers from each coordinate to obtain a point in the initial 2×2 square. Using half angle formulas, the coordinates are

$$x = 60\cos\frac{\pi}{12} = 60\sqrt{\frac{1+\cos\frac{\pi}{6}}{2}} = 30\sqrt{2+\sqrt{3}} \approx 57.95$$

$$y = 60\sin\frac{\pi}{12} = 60\sqrt{\frac{1-\cos\frac{\pi}{6}}{2}} = 30\sqrt{2-\sqrt{3}} \approx 15.53$$

Therefore, the corresponding point in the initial 2×2 square is

$$\left(30\sqrt{2+\sqrt{3}} - 56,\ 30\sqrt{2-\sqrt{3}} - 14\right) \approx (1.95, 1.53).$$

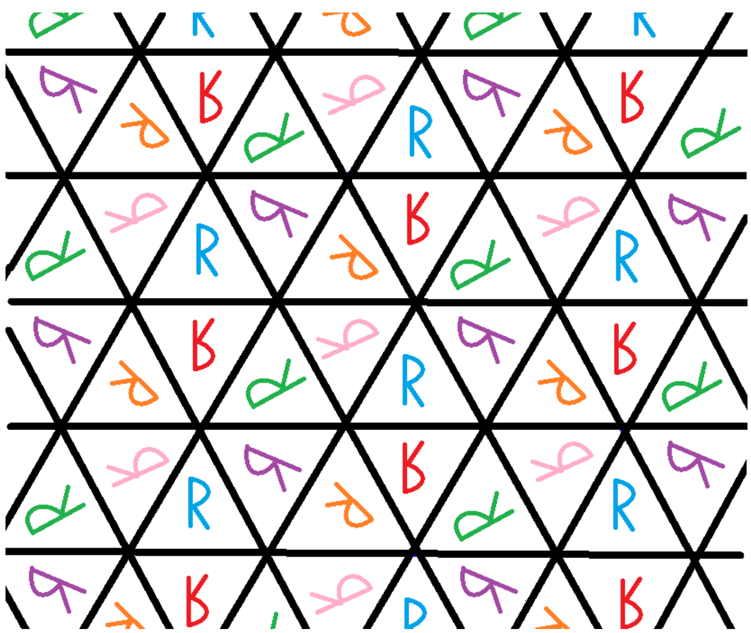
This point is catercorner to the original 1×1 square, so we must flip it across both the vertical line $x = 1$ and horizontal line $y = 1$. This means replacing x with $2 - x$ and y with $2 - y$. Thus, the answer is

$$\left(58 - 30\sqrt{2+\sqrt{3}},\ 16 - 30\sqrt{2-\sqrt{3}}\right).$$

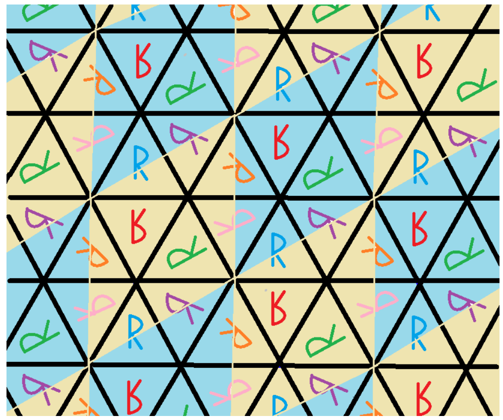
Remark. Another way to evaluate $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$ is using the difference angle formulas, since $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$. In this case, we wind up with the curious "denesting" identities $2\sqrt{2 \pm \sqrt{3}} = \sqrt{6} \pm \sqrt{2}$.

Bonus: Triangular Billiards

With an equilateral triangle, there is likewise a repeating pattern of flipping triangles across edges, however now there are six possible orientations of the letter 'R.'

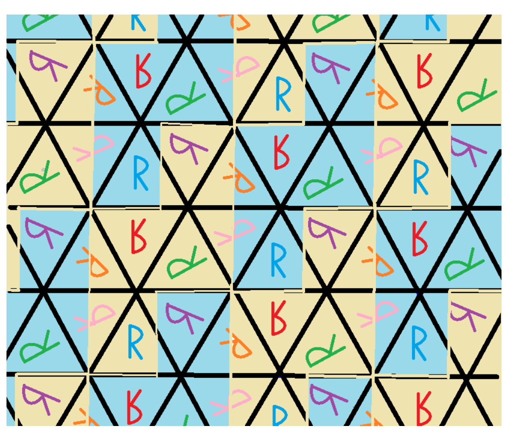


The pattern now repeats in two (non-orthogonal) directions, say along vectors $\vec{u}$ and $\vec{v}$. Their magnitude is double the side length equal to double the height of the triangles. We may draw rhombi whose sides are these two vectors:



By copy-and-pasting this rhombus over and over again, edge to edge, we get a repeating pattern of rhombi. This repeating pattern is called a lattice, tiling, or tessellation. The prototypical tile that all the others are modeled on is called the fundamental polygon. These are studied in e.g. crystallography and hyperbolic geometry.

Any other parallelogram with sides $a\vec{u} + b\vec{v}$ and $c\vec{u} + d\vec{v}$ would also work as a fundamental polygon, so long as the integer matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ has determinant 1 (for this it is necessary but not sufficient that $\gcd(a, b)$ and $\gcd(c, d)$ are both 1). In fact, other non-polygonal shapes may be used as fundamental regions for repeating patterns - for instance, replace the straight edges of a fundamental polygon with curves.



The triangles have base 1 and height $\sqrt{3}/2$, so the vectors are

$$\vec{u} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix}$$

Whatever point the particle would end up at by crossing edges, we may write it in terms of uv-coordinates:

$$\begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} 3/2 & 0 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

To find the components a and b, simply compute

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3/2 & 0 \\ 1 & \sqrt{3} \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}.$$

For the square lattice we could subtract even numbers from either coordinate. For the triangular lattice, we may subtract $\vec{u}$ and $\vec{v}$, which corresponds to subtracting integers from the a and b components (to get their fractional parts) and then converting back to xy-coordinates.

Once this is done, the point will be in one of the handful of triangles overlapping the fundamental rhombus. Compare with the grid points (vertices of triangles) to figure out which triangle it is in and which lines to reflect over, then proceed to reflect it as appropriate until it is in the original triangle.