Problem of the week #2: Solution

For the two-variable version, we may combine like terms:

$$a_0 + a_1(x+y) + a_2(x^2+xy+y^2) + a_3(x^3+x^2y+xy^2+y^3) + \cdots$$

(Arbitrary rearrangement and grouping is legal since we do not need to worry about convergence issues.) This means

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m+n}x^m y^n = \sum_{s=0}^{\infty} a_s(x^s+\cdots+y^s) = \sum_{s=0}^{\infty} a_s\left(\frac{x^{s+1}-y^{s+1}}{x-y}\right)$$

$$= \frac{x\left(\sum_{s=0}^{\infty} a_s x^s\right) - y\left(\sum_{s=0}^{\infty} a_s y^s\right)}{x-y} = \frac{xf(x)-yf(y)}{x-y}.$$

For the four-variable version, notice that the sum of all monomials $w^k x^\ell y^m z^n$ such that $k+\ell=r$ and $m+n=s$, where r and s are fixed, can be factored as (the sum of all monomials $w^k x^\ell$ with $k+\ell=r$) times (the sum of all monomials $y^m z^n$ with $m+n=s$). This is because we may choose the pairs (k,ℓ) and (m,n) independently of each other when deciding on which monomial $w^k x^\ell y^m z^n$ to write down. Therefore

$$\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{k+\ell+m+n} w^k x^\ell y^m z^n$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{r+s}(w^r + \cdots + x^r)(y^s + \cdots + z^s) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{r+s}\left(\frac{w^{r+1} - x^{r+1}}{w - x}\right)\left(\frac{y^{s+1} - z^{s+1}}{y - z}\right)$$

$$= \frac{wy\left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} w^r y^s\right) - wz\left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} w^r z^s\right) - xy\left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} x^r y^s\right) + xz\left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} x^r z^s\right)}{(w-x)(y-z)}$$

$$= \frac{wy\left(\frac{wf(w) - yf(y)}{w - y}\right) - wz\left(\frac{wf(w) - zf(z)}{w - z}\right) - xy\left(\frac{xf(x) - yf(y)}{x - y}\right) + xz\left(\frac{xf(x) - zf(z)}{x - z}\right)}{(w-x)(y-z)}$$

$$= \frac{w^3 f(w)}{(w-x)(w-y)(w-z)} + \frac{x^3 f(x)}{(x-w)(x-y)(x-z)} + \frac{y^3 f(y)}{(y-w)(y-x)(y-z)} + \frac{z^3 f(z)}{(z-w)(z-y)(z-x)}.$$