Problem of the week #2: Background

A polynomial in multiple variables is called homogeneous if every monomial in it has the same total degree. So $x^2 + xy + y^2$ is homogeneous while $x^2 + y$ is not, for example. There is a one-to-one correspondence between polynomials in one variable and homogeneous polynomials in two variables of a given degree. For example,

$$\begin{aligned} x^2 + 1 &\longleftrightarrow & x^2 + y^2 \\ x^2 + 2x + 1 &\longleftrightarrow & x^2 + 2xy + y^2 \\ 4x^3 - 3x &\longleftrightarrow & 4x^3 - 3xy^2 \end{aligned}$$

To turn a degree-n polynomial $f(x)$ into $g(x, y)$, multiply $f(x/y)$ by y^n so all negative powers of y are cancelled. For example

$$y^3 \left[4 \left(\frac{x}{y} \right)^3 - 3 \left(\frac{x}{y} \right) \right] = 4x^3 - 3xy^2.$$

Conversely, to turn $g(x, y)$ into $f(x)$ simply evaluate at $y = 1$. These processes are called **homogenization** and dehomogenization. They may also be applied to rational functions as well.

The geometric sum formula is given by

$$S = 1 + x + \cdots + x^n = \frac{x^n - 1}{x - 1},$$

which follows by comparing S with xS. Homogenizing yields

$$x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n = \frac{x^{n+1} - y^{n+1}}{x - y}.$$