Problem of the week #1: Solutions

$$S(t) = \int_0^t \frac{\mathrm{d}\tau}{\cosh\tau}, \qquad T(s) = \int_0^s \frac{\mathrm{d}\sigma}{\cos\sigma}.$$

The function $S(t)$ is called the **Gudermannian function** and $T(s)$ the inverse Gudermannian function. Writing $S = S(T)$ or $T = T(S)$, they satisfy the following three equivalent identities:

$$\begin{array}{ll} (1) & \tan S = \sinh T \\ (2) & \sin S = \tanh T \\ (3) & \cos S = \operatorname{sech} T \end{array}$$

For example, (1) says $\tan S(t) = \sinh t$ and $\tan s = \sinh T(s)$.

One identity may be converted into another by applying transformations, using circular Pythagorean identities on the left and hyperbolic Pythagorean identities on the right. Applying $\sqrt{1-x^2}$ converts between (2) and (3); applying $x/\sqrt{1+x^2}$ converts from (1) to (2) and its inverse $x/\sqrt{1-x^2}$ from (2) to (1); applying $1/\sqrt{1+x^2}$ converts from (1) to (3) and its inverse $\sqrt{1-x^2}/x$ from (3) to (1). When converting from (3) it suffices to assume $S, T \geq 0$ since they are odd functions.

There is also a fourth equivalent half-angle identity

$$(4) \qquad \tan(S/2) = \tanh(T/2).$$

This follows from all (hence any) of (1),(2),(3) using either version of tan and tanh's half-angle formulas. For example,

$$\tan\frac{S}{2} = \frac{\sin S}{1+\cos S} = \frac{\tanh T}{1+\operatorname{sech} T} = \frac{\sinh T}{\cosh T + 1} = \tanh\frac{T}{2}.$$

Conversely, (4) may be converted to (1) by applying $2x/(1-x^2)$, to (2) by applying $2x/(1+x^2)$, and to (3) by applying $(1-x^2)/(1+x^2)$; therefore all of the identities (1),(2),(3),(4) are equivalent to each other.

To show $S(t)$ and $T(s)$ are inverse functions, it suffices to establish any of the four identities for $S(t)$ and t and any other one of the four for s and $T(s)$. For example, if evaluating $S(t)$ yields (1) and evaluating $T(s)$ yields (2), then (1) implies (2) so $S(t) = \sin^{-1}(\tanh t)$ and $T(s) = \tanh^{-1}(\sin s)$ and hence they are inverse functions.

Cofunction substitutions. Evaluate the definite integral $S(t)$ using the substitution $u = \sinh(\tau)$ (where $\mathrm{d}u = \cosh(\tau)\mathrm{d}\tau$) and the hyperbolic Pythagorean identity $\cosh^2 - \sinh^2 = 1$:

$$S(t) = \int_0^t \frac{\cosh(\tau)\mathrm{d}\tau}{1+\sinh^2(\tau)} = \int_0^{\sinh t} \frac{\mathrm{d}u}{1+u^2} = \tan^{-1}(\sinh(t)).$$

Evaluate $T(s)$ first by using the substitution $u = \sin(\sigma)$ (where $\mathrm{d}u = \cos(\sigma)\mathrm{d}u$) and the circular Pythagorean identity $\cos^2 + \sin^2 = 1$:

$$T(s) = \int_0^s \frac{\cos(\sigma)\mathrm{d}\sigma}{1-\sin^2(\sigma)} = \int_0^{\sin s} \frac{\mathrm{d}u}{1-u^2} = \tanh^{-1}(\sin(s)).$$

Without directly knowing or recognizing the derivative of $\tanh^{-1}$, it is also possible to use partial fraction decomposition:

$$\int_0^{\sin s} \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)\mathrm{d}u = \frac{1}{2}\ln\left|\frac{1+\sin s}{1-\sin s}\right| = \tanh^{-1}(\sin(s)).$$

Exponential substitutions. Evaluate $S(t)$ using the substitution $u = e^\tau$ (where $\mathrm{d}u = e^\tau \mathrm{d}\tau$) and absorbing 2 into the integral by doubling the interval over which it is taken (since cosh is an even function):

$$S(t) = \int_0^t \frac{2\mathrm{d}\tau}{e^\tau + e^{-\tau}} = \int_{-t}^t \frac{e^\tau \mathrm{d}\tau}{e^{2\tau}+1} = \int_{e^{-t}}^{e^t} \frac{\mathrm{d}u}{u^2+1} = \tan^{-1}(e^t) - \tan^{-1}(e^{-t}).$$

Apply tangent with difference-angle identity to get

$$\tan S(t) = \frac{e^t - e^{-t}}{1 + e^t e^{-t}} = \sinh(t).$$

We may instead have chosen to divide S by 2 in which case

$$\frac{S(t)}{2} = \int_0^t \frac{\mathrm{d}\tau}{e^\tau + e^{-\tau}} = \int_1^{e^t} \frac{\mathrm{d}u}{u^2+1} = \tan^{-1}(e^t) - \tan^{-1}(1).$$

Applying tangent (and multiplying by $e^{-t/2}/e^{-t/2}$) yields

$$\tan\frac{S(t)}{2} = \frac{e^t - 1}{1 + e^t} = \frac{(e^{t/2} - e^{-t/2})/2}{(e^{t/2} + e^{-t/2})/2} = \tanh(t/2).$$

Phasor substitutions. With functions of complex variables and path integrals in the complex plane it is possible to evaluate $T(s)$ using the substitution $u = e^{i\sigma}$ (where $\mathrm{d}u = ie^{i\sigma}\mathrm{d}\sigma$) alongside the formula $\cos(\sigma) = (e^{i\sigma} + e^{-i\sigma})/2$. Absorbing 2 into the integral yields:

$$T(s) = \int_0^s \frac{2\mathrm{d}\sigma}{e^{i\sigma} + e^{-i\sigma}} = \int_{-s}^s \frac{e^{i\sigma}\mathrm{d}\sigma}{e^{2i\sigma} + 1} = \frac{1}{i}\int_{e^{-is}}^{e^{is}} \frac{\mathrm{d}u}{u^2 + 1}$$

$$= \frac{\tan^{-1}(e^{is}) - \tan^{-1}(e^{-is})}{i}.$$

Apply tangent with difference angle identity to get

$$\tan iT(s) = \frac{e^{is} - e^{-is}}{1 + e^{is}e^{-is}} = i \sin s,$$

which is equivalent to $\tanh T(s) = \sin s$. Dividing by 2 instead,

$$\frac{T(s)}{2} = \int_0^s \frac{d\sigma}{e^{i\sigma} + e^{-i\sigma}} = \frac{1}{i}\int_0^{e^{is}} \frac{\mathrm{d}u}{u^2 + 1} = \frac{\tan^{-1}(e^{is}) - \tan^{-1}(1)}{i}$$

Applying tangent (and multiplying by $e^{-is/2}/e^{-is/2}$) yields

$$\tan \frac{iT(s)}{2} = \frac{e^{is} - 1}{1 + e^{is}} = \frac{e^{is/2} - e^{is/2}}{e^{is/2} + e^{-is/2}} = \tan(s/2).$$

Differentiation. Another idea: we may show $(T \circ S)(t) = t$ by showing both sides have the same derivative and agree at the initial value $(T \circ S)(0) = 0$. Differentiating with the chain rule and solving for S indicates we need to show $S(t) = \pm \cos^{-1}(\mathrm{sech}(t))$. This, again, can be argued by showing both sides are equal at $t = 0$ and have the same derivative, though one needs to manage the continuity of the $\pm$ sign, and then the same can be done to show $(S \circ T)(s) = s$, or else argue S and T are one-to-one because they are monotonic because their integrands are always positive on S and T's domains.