**Problem of the week #1**: Background

Euler's formula $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$ may be justified a few ways:

- Plug $i\theta$ into the Taylor series for the natural exponential function. Notice the powers of $i$ repeat every four terms, and alternate between real and imaginary. Then notice the real and imaginary parts are actually the Taylor series for cos and sin.

- Define $\exp(i\theta)$ as the solution to $f'(\theta) = if(\theta)$, where $\theta$ is a real variable but $f$ is complex-valued. Since $i$ rotates $90^\circ$ in the complex plane, the derivative is tangential velocity of a constant-speed circular motion, so $f(\theta)$ goes around the unit circle in the complex plane assuming the initial condition $f(0) = 1$.

- In the "imaginary" compound interest formula

$$\exp(i\theta) = \lim_{n\to\infty} \left(1 + \frac{i\theta}{n}\right)^n,$$

  heuristically (i.e. $\sin\varepsilon \approx \varepsilon$ when $\varepsilon \approx 0$) we may interpret $1 + i\theta/n$ as approximately a point on the circle an angle of $\theta/n$ from 1, so the $n$th power ought to be approximately a point an angle $\theta$ from 1. (Showing the error tends to 0 is necessary for a proof.)

By "symmetrizing" or "antisymmetrizing" Euler's formula we get

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

The unit circle $x^2 + y^2 = 1$ may be parametrized with trigonometric functions using $(\cos\theta, \sin\theta)$. The unit hyperbola $x^2 - y^2 = 1$ may have its right half parametrized with so-called *hyperbolic* trigonometric functions using $(\cosh t, \sinh t)$, where we define

$$\cosh(t) = \frac{e^t + e^{-t}}{2}, \qquad \sinh(t) = \frac{e^t - e^{-t}}{2}.$$

To see this, notice we may rotate $x^2 - y^2 = 1$ by $45^\circ$ (and scale by $\sqrt{2}$) using the diagonal coordinates $u = x + y$ and $v = x - y$ in which case the equation becomes $uv = 1$. Positive reals may be parametrized with exponentials, so writing $u = e^t$ and $v = e^{-t}$ we may solve for $x$ and $y$ to get the formulas for cosh and sinh above.

The four other hyperbolic trig functions $\tanh, \coth, \operatorname{sech}, \operatorname{csch}$ are defined in terms of cosh and sinh analogously to how $\tan, \cot, \sec, \csc$ are defined in terms of cos and sin. Oftentimes identities for circular trig functions may be converted to ones for hyperbolic trig functions using

$$\cos(ix) = \cosh(x), \quad \sin(ix) = i\sinh(x), \quad \tan(ix) = i\tanh(x).$$

For instance, the substitution $t = i\theta$ yields

$$\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta},$$

$$\tanh\left(\frac{t}{2}\right) = \frac{\cosh t - 1}{\sinh t} = \frac{\sinh t}{\cosh t + 1}.$$

Hyperbolic trig functions also enjoy Pythagorean identities:

$$\begin{aligned} \cos^2(x) + \sin^2(x) &= 1 & \cosh^2(x) - \sinh^2(x) &= 1 \\ 1 + \tan^2(x) &= \sec^2(x) & 1 - \tanh^2(x) &= \operatorname{sech}^2(x) \\ \cot^2(x) + 1 &= \csc^2(x) & \coth^2(x) - 1 &= \operatorname{csch}^2(x) \end{aligned}$$

They have derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh(x) = \sinh(x), \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sinh(x) = \cosh(x)$$

and their inverses may be solved explicitly using the natural logarithm; for instance to find $\tanh^{-1}$ set up and then solve

$$x = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

for $y$ first by clearing the denominator and multiplying by $e^y$ to get

$$\tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$