Problem of the week #11: Solution

2D Constructions. There are certain compass and straightedge constructions that can also be generalized to 3D and will be useful for our solution. We consider these as a warm-up.

One may construct the line through a point P which is perpendicular to the line L between points P and Q as follows. First, draw a circle C around P through Q, call the intersection between C and L the point Q'. Draw a circle D through Q' around Q and another circle D' through Q around Q', and let R be one of the intersection points between D and D'. Finally, draw the line between P and R.

Similarly, one can construct the midpoint of a line segment between points P and Q as follows. Draw a circle C around P through Q and another circle D around Q through P. Call the intersection points between circles C and D the points R and S. Draw the line L between R and S, and then the intersection M between the line L and the line segment from P to Q will be the midpoint of P and Q.

Given a known plane in 3D, any 2D construction may be performed in that plane using astrolabe and flatedge by intersecting with the plane at each step. For example, given two points C and X in the plane, construct the sphere around C through X, the intersect that sphere with the plane to get the circle around C through X within that plane.

Constructing Three Perpendicular Lines.

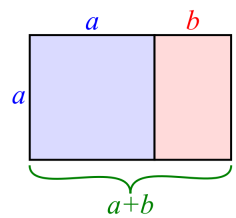
Given a point C, pick any two other points W and X and form the plane P between the three points C, W, X with the flatedge tool. Within the plane P, draw the line L from C to X and construct the perpendicular line M through C. Draw the sphere S around C through X and intersect with the line M to get another point Y. Say the intersection of S with line L consists of points X, X' and the intersection of S with line M consists of points Y, Y'. We want two points Z, Z' on an axis N perpendicular to L and M.

Form spheres around X through X' and vice versa and intersect to get a circle C in the YZ-plane perpendicular to L. Similarly, form spheres and Y through Y' and vice-versa and intersect to get a circle D in the XZ-plane perpendicular to M. The circles C and D intersect desired points Z, Z' through which we may form the line N. Note in this picture Z, Z' are further from the center C than X, X', Y, Y'.

The three lines L, M, N are three perpendicular lines through C.

Constructing Golden Rectangles.

A rectangle is called *golden* if the smaller rectangle alongside an inscribed square is similar to (same proportions as) the whole rectangle:



The equation $(a+b)/a = a/b$ becomes $1+1/x = x$ if we define $x = a/b$, and solving the subsequent quadratic equation $x^2 - x - 1 = 0$ yields the golden ratio $\varphi = (1+\sqrt{5})/2$. Thus, a rectangle is golden if the proportion a/b between its sides is the golden ratio.

To construct a golden rectangle with a given center C in a plane, it suffices to construct four equal-size golden rectangles around it. Thus, given perpendicular lines L and M intersecting at a corner C in a plane, it suffices to be able to construct a golden rectangle with a given line segment $\overline{CX}$ along line L. By fiat declare this to be unit length.

Draw a circle around C through X and intersect with line M to get two points Y, Y'. Construct a line L' perpendicular to L through X, and another line M' perpendicular to M through Y. Call C' the intersection of lines L' and M'. Thus $\square CXC'Y$ is a unit square.

Next, draw a circle around Y' through C' and intersect with M to get a point W on the other side of Y' from C. Draw a circle around C through W and again intersect with M to get another point W'. Note W and W' are both a distance of $\sqrt{5}$ from C, by the Pythagorean theorem applied to the right triangle $\triangle C'YY'$.

Finally, construct the midpoint G of the line segment $\overline{CW'}$. Construct a line M'' perpendicular to M at G. The lines L' and M'' intersect at a point, say H. Then $CXGH$ is a golden rectangle.

Constructing Icosaheda.

Finally, after constructing three perpendicular lines intersecting at a point C, pick a point X on one of them, form a sphere around C through X and intersect with the three lines to get pairs X, X' and Y, Y' and Z, Z' on the three axes. And may use these line segments to construct three golden rectangles in the three corresponding planes.

By drawing line segments between neighboring corners of the golden rectangles we obtain a regular icosahedron.