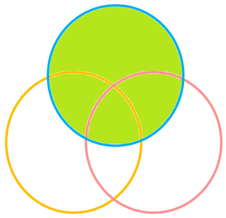
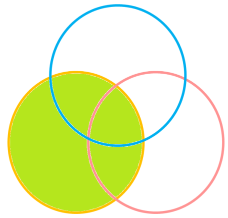
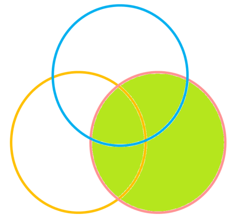
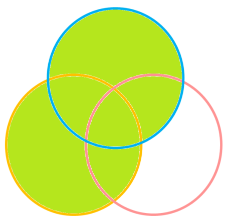
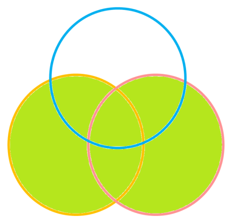
Problem of the week #10: Solution

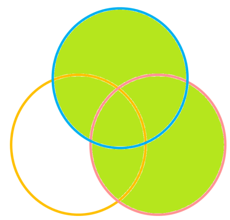

A

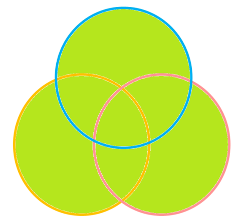

B

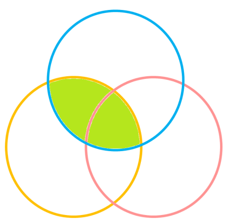

C

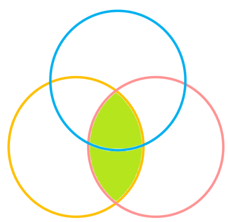

$A \cup B$

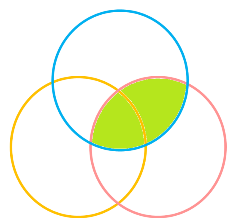

$B \cup C$

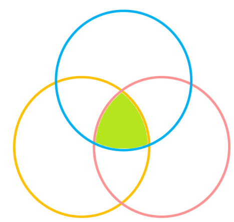

$C \cup A$

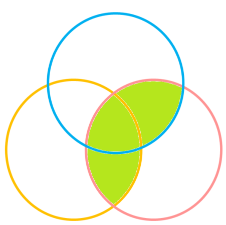

$A \cup B \cup C$

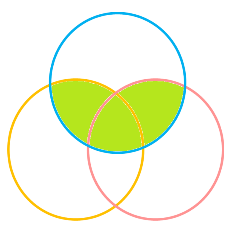

$A \cap B$

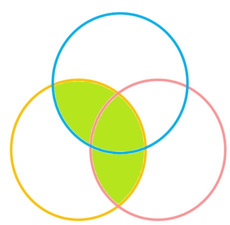

$B \cap C$

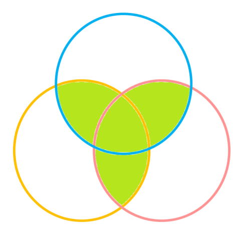

$C \cap A$

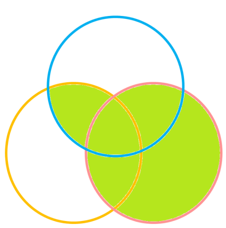

$A \cap B \cap C$

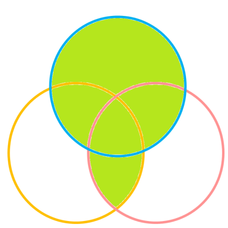

$(A \cup B) \cap C$

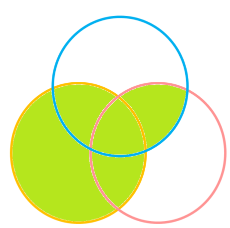

$(B \cup C) \cap A$


$(C \cup A) \cap B$


X


$(A \cap B) \cup C$

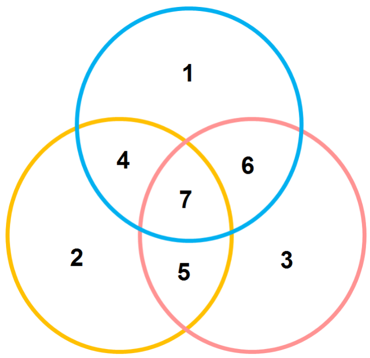

$(B \cap C) \cup A$


$(C \cap A) \cup B$

Note the last set in the lower right corner has two representations:

$$X = (A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$$

All of these diagrams can be guaranteed distinct when each of the seven possible regions in the circles is nonempty. For instance, let A, B, C literally be the circles depicted in the plane. Or label each of the regions 1-7 and construct the corresponding sets A, B, C:



$$A = \{1, 4, 6, 7\}$$

$$B = \{2, 4, 5, 7\}$$

$$C = \{3.5.6.7\}$$

It can be manually checked that unioning or intersecting any two of the listed sets yields another one of these sets. Therefore, the maximal number of distinct sets is 18.

If counting empty union and empty intersection, there are 20 sets.

The empty union is the union of no sets, which must be the empty set because that is the only set that does affect anything else when unioning. Similarly, the empty intersection must be the entire universe $\mathcal{U}$ under consideration. In a Venn diagram, this would be all three circles filled in, plus the rest of a rectangle outside of it.