Solution to Problems ♣–6789

Problem A: *Given* $q = x + yi + zj + wk$*, let* $\bar{q} = x - yi - zj - wk$ *be its conjugate. Verify that the Euclidean norm*

$$\|q\| := \sqrt{x^2 + y^2 + z^2 + w^2} = \sqrt{q\bar{q}}.$$

Proof. By straightforward computation. □

Problem B: *We identify* $\mathbb{R}^3$ *with the subspace of* H *spanned by* $\{i, j, k\}$*. Given two vectors* v *and* v' *in* $\mathbb{R}^3$*, verify that*

$$v\bar{v}' = \langle v, v' \rangle - v \times v',$$

where $\langle , \rangle$ *is the Euclidean inner product (or dot product) and* $\times$ *is the cross product on* $\mathbb{R}^3$*.*

Proof. By straightforward computation. □

Problem C: *For each* $q \in U$*, we define*

$$f_q : \mathbb{R}^3 \to \mathbb{R}^3 \text{ with } f_q(v) = qv\bar{q}.$$

Verify that f_q *is a linear map of* $\mathbb{R}^3$*. What is* f_q^{-1}*?*

Proof. For the first part, simply use the definition of a linear map. For the second part, we have $f_q^{-1} = f_{\bar{q}}$. □

Problem D: *Given $q \in U$, show that the linear map f_q defined above is a 3-dimensional rotation. A rotation on $\mathbb{R}^3$ is specified by a rotation axis and a rotation angle. Can you find the axis and the angle of the rotation for f_q? Does your answer for f_q^{-1} make sense when view them as rotations?*

Proof. We first consider an easy case and let $q = \cos(\theta) + \sin(\theta)i$ in U. You can compute how f_q acts on each basis element $\{i, j, k\}$ of $\mathbb{R}^3$:

$$\begin{aligned} f_q(i) &= i \\ f_q(j) &= \cos(2\theta)j + \sin(2\theta)k \\ f_q(k) &= -\sin(2\theta)j + \cos(2\theta)k. \end{aligned}$$

So with respect to the basis $\{i, j, k\}$, the linear map $f_q : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix

$$(1) \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\theta) & -\sin(2\theta) \\ 0 & \sin(2\theta) & \cos(2\theta) \end{pmatrix}.$$

Hence, f_q is a rotation. The axis is the line going through the origin in the vector i direction, and the rotation angle is 2θ. You can see this by noticing that f_q fixes i and the submatrix $\begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix}$ rotates the plane spanned by $\{j, k\}$ about the origin by the angle 2θ.

Now we consider the general case. Let $q = x + yi + zj + wk \in U$. We are going to find a suitable basis of $\mathbb{R}^3$ so that under that basis, the linear map f_q is also represented by the matrix as in (1).

Since $\|q\| = 1$, we can rewrite

$$q = \cos(\theta) + \sin(\theta)v_1$$

where v_1 is a unit vector in $\mathbb{R}^3$. In particular, we have $x = \cos(\theta)$ and you can figure out what v_1 needs to be in terms of y, z, w and $\sin(\theta)$.

Let v_2 be a unit vector that is orthogonal to v_1. Let $v_3 = v_1v_2$, which is also a unit vector. We claim that

(a) $\{v_1, v_2, v_3\}$ forms a orthonormal basis of $\mathbb{R}^3$.
(b) The multiplication table for $\{v_1, v_2, v_3\}$ is exactly the same as the table for $\{i, j, k\}$.

Claim (b) is an immediate consequence of Exercise (1) or (2). For example,

$$v_1v_2 = v_3 \Rightarrow v_1v_2v_3 = -1 \Rightarrow v_2v_3 = v_1.$$

To show (a), you will use Exercise (2) to verify that v_3 is orthogonal to both v_1 and v_2. We omit the details here.

Finally based on claims (a) and (b), replacing i by v_1, j by v_2 and k by v_3, the exact computation we did for the special case above proves that with respect to the new basis $\{v_1, v_2, v_3\}$, the linear map f_q is also represented by the matrix in (1). Therefore, for $q = x + yi + zj + wk = \cos(\theta) + \sin(\theta)v_1$, the linear map $f_q : v \mapsto qv\bar{q}$ is a rotation with axis v_1 and rotation angle 2θ.

□

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