# Solution to Problem ♣–5

**Problem:** *Show that for every positive real numbers* $a_1, \dots, a_n$ *the following inequality holds true:*

$$\sum_{i=1}^{n} \frac{1}{a_i} \geq \frac{n^2}{\sum\limits_{i=1}^{n} a_i}$$

*Proof.* We are showing the statement in the problem by induction on $n$. First we note that if $n = 1$ and $a_1 > 0$ then

$$\sum_{i=1}^{n} \frac{1}{a_i} = \frac{1}{a_1} \geq \frac{n^2}{\sum\limits_{i=1}^{n} a_i}$$

(so the initial step of the inductive proof is verified).

Suppose now that the inequality in question holds for any $n$ positive real numbers. Assume that $a_1, \dots, a_n, a_{n+1}$ are positive real numbers. Then, by the inductive hypothesis, we have

$$\sum_{i=1}^{n} \frac{1}{a_i} \geq \frac{n^2}{\sum\limits_{i=1}^{n} a_i},$$

and hence

$$\sum_{i=1}^{n+1} \frac{1}{a_i} = \sum_{i=1}^{n} \frac{1}{a_i} + \frac{1}{a_{n+1}} \geq \frac{n^2}{\sum\limits_{i=1}^{n} a_i} + \frac{1^2}{a_{n+1}}.$$

In Problem ♣–4 (which was due on 09/21/18) we showed that for every real numbers $a, b, c, d$ such that $c, d > 0$ we have

$$\frac{(a+b)^2}{c+d} \leq \frac{a^2}{c} + \frac{b^2}{d}.$$

Hence

$$\frac{n^2}{\sum\limits_{i=1}^{n} a_i} + \frac{1^2}{a_{n+1}} \geq \frac{(n+1)^2}{(\sum\limits_{i=1}^{n} a_i) + a_{n+1}} = \frac{(n+1)^2}{\sum\limits_{i=1}^{n+1} a_i}.$$

This completes the proof of the inductive step of our arguments. □

Correct solution was received from :

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