Solutions to Problems ♣–14/15

Problem14: *Let $\theta > 0$. Show that*

$$\frac{\sin(\theta)}{\theta} = \prod_{n=1}^{\infty} \cos\left(\frac{\theta}{2^n}\right).$$

Solution. In Problem ♣–13 (which was due on 11/16/18) we showed that for any positive integer n and $\theta > 0$ we have

$$\sin(\theta) = 2^n \cdot \sin\left(\frac{\theta}{2^n}\right) \cdot \cos\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{4}\right) \cdot \cos\left(\frac{\theta}{8}\right) \cdot \ldots \cdot \cos\left(\frac{\theta}{2^n}\right).$$

Consequently,

$$(\circledast) \qquad \frac{\sin(\theta)}{\theta} \cdot \frac{\theta/2^n}{\sin\big(\theta/2^n\big)} = \cos\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{4}\right) \cdot \cos\left(\frac{\theta}{8}\right) \cdot \ldots \cdot \cos\left(\frac{\theta}{2^n}\right).$$

Since $\lim\limits_{x\to 0} \frac{\sin(x)}{x} = 1$, we know

$$\lim_{n\to\infty} \frac{\theta/2^n}{\sin\big(\theta/2^n\big)} = 1.$$

Therefore, letting $n \longrightarrow \infty$ in the eqution $(\circledast)$ above, we conclude

$$\frac{\sin(\theta)}{\theta} = \lim_{n\to\infty} \Big(\cos\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{4}\right) \cdot \cos\left(\frac{\theta}{8}\right) \cdot \ldots \cdot \cos\left(\frac{\theta}{2^n}\right)\Big).$$

In other words, $\frac{\sin(\theta)}{\theta} = \prod\limits_{n=1}^{\infty} \cos\left(\frac{\theta}{2^n}\right)$, as required. □

CORRECT SOLUTION WAS RECEIVED FROM :

(1) GAGE HOEFER	POW 14: ♣
(2) GRANT MOLES	POW 14: ♣
(3) WOFGANG L. NORGREN	POW 14: ♣
(4) FREDERIC PENNEY	POW 14: ♣
(5) HENRIK PENNEY	POW 14: ♣
(6) ZACH SABATA	POW 14: ♣
(7) DAYNE SHIELDS	POW 14: ♣
(8) BRAD TUTTLE	POW 14: ♣

Problem15: *Show that*

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \cdot \ldots$$

Solution. First note that, for any real x, we have $\cos(2x) = 2\cos^2(x) - 1$. Hence, for any $n \in \mathbb{N}$,

$$\cos\left(2 \cdot \frac{\pi/2}{2^n}\right) = 2\cos^2\left(\frac{\pi/2}{2^n}\right) - 1,$$

and thus

$$\cos\left(\frac{\pi}{2^{n+1}}\right) = \sqrt{\frac{\cos\left(\frac{\pi}{2^n}\right) + 1}{2}}.$$

Consequently,

$$\prod_{n=1}^{\infty} \cos\left(\frac{\pi}{2^{n+1}}\right) =$$

$$\cos(\pi/4) \cdot \sqrt{\frac{\cos(\pi/4)+1}{2}} \cdot \sqrt{\frac{\sqrt{\frac{\cos(\pi/4)+1}{2}}+1}{2}} \sqrt{\frac{\sqrt{\frac{\sqrt{\frac{\cos(\pi/4)+1}{2}}+1}{2}}+1}{2}} \ldots =$$

$$\frac{\sqrt{2}}{2} \cdot \sqrt{\frac{\frac{\sqrt{2}}{2}+1}{2}} \cdot \sqrt{\frac{\sqrt{\frac{\frac{\sqrt{2}}{2}+1}{2}}+1}{2}} \cdot \sqrt{\frac{\sqrt{\frac{\sqrt{\frac{\frac{\sqrt{2}}{2}+1}{2}}+1}{2}}+1}{2}} \cdot \ldots =$$

$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \sqrt{\frac{\frac{\sqrt{2+\sqrt{2}}}{2}+1}{2}} \cdot \sqrt{\frac{\sqrt{\frac{\frac{\sqrt{2+\sqrt{2}}}{2}+1}{2}}+1}{2}} \cdot \ldots$$

$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \sqrt{\frac{\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}+1}{2}} \cdot \ldots$$

$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \cdot \ldots$$

Finally, by Problem ♣–14 above we have

$$\frac{\sin(\pi/2)}{\pi/2} = \prod_{n=1}^{\infty} \cos\left(\frac{\pi/2}{2^n}\right),$$

so together

$$\frac{2}{\pi} = \frac{\sin(\pi/2)}{\pi/2} = \prod_{n=1}^{\infty} \cos\left(\frac{\pi}{2^{n+1}}\right) =$$

$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \cdot \ldots$$

□

Correct solution was received from :

(1) Gage Hoefer POW 15: ♣
(2) Grant Moles POW 15: ♣
(3) Zach Sabata POW 15: ♣
(4) Dayne Shields POW 15: ♣
(5) Brad Tuttle POW 15: ♣