Solution to Problem ♣–13

Problem: *Show that for any positive integer n and $\theta > 0$ we have*

$$\sin(\theta) = 2^n \cdot \sin\left(\frac{\theta}{2^n}\right) \cdot \cos\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{4}\right) \cdot \cos\left(\frac{\theta}{8}\right) \cdot \ldots \cdot \cos\left(\frac{\theta}{2^n}\right).$$

Solution. We show this statement by induction on $n \in \mathbb{N}$. Let $\Phi(n)$ be the assertion

$$\Big(\forall\theta > 0\Big)\Big(\sin(\theta) = 2^n \cdot \sin\left(\frac{\theta}{2^n}\right) \cdot \cos\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{4}\right) \cdot \cos\left(\frac{\theta}{8}\right) \cdot \ldots \cdot \cos\left(\frac{\theta}{2^n}\right)\Big).$$

We will verify that Φ satisfies the assumptions of the Theorem on Mathematical Induction.

Basic Step: It is well known that for each θ we have

$$\sin(\theta) = \sin\left(2 \cdot \frac{\theta}{2}\right) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right).$$

Consequently, $\Phi(1)$ holds true.

Inductive Step: We are going to argue that

$$\big(\forall n \in \mathbb{N}\big)\big(\Phi(n) \;\Rightarrow\; \Phi(n+1)\big)$$

holds true. To this end, suppose $n \in \mathbb{N}$ and assume that $\Phi(n)$ holds true. Suppose $\theta > 0$. Applying our inductive hypothesis $\Phi(n)$ we get

$$\sin(\theta) = 2^n \cdot \sin\left(\frac{\theta}{2^n}\right) \cdot \cos\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{4}\right) \cdot \cos\left(\frac{\theta}{8}\right) \cdot \ldots \cdot \cos\left(\frac{\theta}{2^n}\right).$$

However,

$$\sin\left(\frac{\theta}{2^n}\right) = \sin\left(2 \cdot \frac{\theta}{2^{n+1}}\right) = 2\sin\left(\frac{\theta}{2^{n+1}}\right)\cos\left(\frac{\theta}{2^{n+1}}\right),$$

so we get

$$\begin{aligned}
&\sin(\theta) = \\
&2^n \cdot 2\sin\left(\tfrac{\theta}{2^{n+1}}\right)\cos\left(\tfrac{\theta}{2^{n+1}}\right) \cdot \cos\left(\tfrac{\theta}{2}\right) \cdot \cos\left(\tfrac{\theta}{4}\right) \cdot \cos\left(\tfrac{\theta}{8}\right) \cdot \ldots \cdot \cos\left(\tfrac{\theta}{2^n}\right) = \\
&2^{n+1}\sin\left(\tfrac{\theta}{2^{n+1}}\right) \cdot \cos\left(\tfrac{\theta}{2}\right) \cdot \cos\left(\tfrac{\theta}{4}\right) \cdot \cos\left(\tfrac{\theta}{8}\right) \cdot \ldots \cdot \cos\left(\tfrac{\theta}{2^n}\right) \cdot \cos\left(\tfrac{\theta}{2^{n+1}}\right).
\end{aligned}$$

Consequently, $\Phi(n+1)$ holds true. □

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