Problem: Assume that Earth is a sphere with radius $R$. (In reality, $R$ is about 6400 kilometers, but this has nothing to do with the solution of this problem. Express everything, including your answer, in terms of $R$.) A satellite has an elliptical orbit with the centre of Earth at one focus. The lowest point of the orbit is $5R$ above the surface of Earth, when the satellite is directly above the North Pole. The highest point of the orbit is $11R$ above the surface of Earth, when the satellite is directly above the South Pole. What is the height of the satellite above the surface of Earth, when the satellite is directly above the equator?

Solution. Let the center of Earth (and a focus of the ellipse) be at the point $(0, c)$, $c > 0$. Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$  

It is given that the vertices of the ellipse on the $y$-axis are $(0, (c + R) + 5R)$ and $(0, (c - R) - 11R)$. It follows that the length of the major axis on the $y$-axis is $2b = 6R + 12R = 18R$. Thus, $b = 9R$ and $c = b - 6R = 3R$. From $c^2 = b^2 - a^2$ we get that $a^2 = 72R^2$. Thus the equation of the ellipse is

$$\frac{x^2}{72R^2} + \frac{y^2}{81R^2} = 1.$$  

The question is to evaluate the value of $|x|$ when $y = c = 3R$. From

$$\frac{x^2}{72R^2} + \frac{9R^2}{81R^2} = 1.$$  

it follows that $|x| = 8R$. Consequently, the height of the satellite above the surface of Earth when it is directly above the equator is $7R$. □

Correct solution was received from:

1. Grant Moles
2. Henrik Penney
3. Roman Penney
4. Brad Tuttle