Solution to Problem ♣–1

Problem: *Prove that for every $x \in \mathbb{R}$ we have*

$$e^x \le e^{x^2} + x.$$

Proof. Let $f(x) = e^{x^2} + x - e^x$ for $x \in \mathbb{R}$. Note that f is differentiable infinitely many times and

- $f'(x) = 2xe^{x^2} - e^x + 1$,
- $f''(x) = e^{x^2}(2 + 4x^2) - e^x$.

The function $g(x) = x^2 - x$ attains the minimum at $x_0 = 1/2$, so the function $h(x) = e^{x^2 - x}$ attains the minimum at $x_0 = 1/2$ and the smallest value is $e^{-1/4} > 1/2$. Hence, for every $x \in \mathbb{R}$,

$$e^{x^2 - x} \cdot (2 + 4x^2) \ge 2e^{-1/4} > 1,$$

and

$$e^{x^2} \cdot (2 + 4x^2) > e^x.$$

Consequently, $f''(x) > 0$ for all $x \in \mathbb{R}$ and the function f' must be strictly increasing. Since $f'(0) = 0$ we conclude that

- $f'(x) < 0$ for all $x < 0$, and
- $f'(x) > 0$ for all $x > 0$.

Therefore,

- f is strictly decreasing on $(-\infty, 0]$ and
- f is strictly increasing on $[0, \infty)$.

Since $f(0) = 0$ necessarily $f(x) \ge 0$ for all $x \in \mathbb{R}$ and the desired inequality follows. □

There were 8 solutions submitted. Correct solution was received from :

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(2) Brad Tuttle — POW 1: ♣