Problems ♣-6789

Due in DSC 222 by 12 noon, **Friday, October 19**, *2018*

Each problem is worth 1 point and you have up to 3 weeks to solve them!

The problems were prepared by Professor Ying Hu and she will be grading your solutions too. If you are interested in the topic or have any additional questions, you may contact Professor Ying Hu at

yinghu@unomaha.edu

* *This set of problems means to give you a hint on how the group of unit quaternions, denoted by U below, acts on the Euclidean space $\mathbb{R}^3$ by isometries. This defines a surjective group homormophism $\rho : U \to SO(3)$ with kernel $\mathbb{Z}_2$. On the other hand, both U and $SO(3)$ are also topological spaces. From the viewpoint of topology, group homomorphism ρ possesses very nice topological properties, which make it a covering map with fiber $\mathbb{Z}_2$.*

* *You will need to know some basic linear algebra in order to complete the problem set.*

The algebra of quaternions

$$H = \{x + yi + zj + wk : x, y, z, w \in \mathbb{R}\}$$

as a vector space is spanned by $\{1, i, j, k\}$ over $\mathbb{R}$. Its multiplication is determined by the multiplication table below

*	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

For instance, $(1 + 2j - k)(1 + i) = 1 + 2j - k + i + 2ji - ki = 1 + i + j - 3k$.

Problem A: Given $q = x + yi + zj + wk$, let $\bar{q} = x - yi - zj - wk$ be its conjugate. Verify that the Euclidean norm

$$\|q\| := \sqrt{x^2 + y^2 + z^2 + w^2} = \sqrt{q\bar{q}}.$$

Problem B: We identify $\mathbb{R}^3$ with the subspace of H spanned by $\{i, j, k\}$. Given two vectors v and v' in $\mathbb{R}^3$, verify that

$$v\bar{v}' = \langle v, v' \rangle - v \times v',$$

where $\langle , \rangle$ is the Euclidean inner product (or dot product) and $\times$ is the cross product on $\mathbb{R}^3$.

The 3-dimensional sphere $S^3 = \{v \in \mathbb{R}^4 : \|v\| = 1\}$ is naturally identified with the group of quaternions of norm 1:

$$U = \{x + yi + zj + wk \in H : x^2 + y^2 + z^2 + w^2 = 1\}.$$

Problem C: For each $q \in U$, we define

$$f_q : \mathbb{R}^3 \to \mathbb{R}^3 \text{ with } f_q(v) = qv\bar{q}.$$

Verify that f_q is a linear map of $\mathbb{R}^3$. What is f_q^{-1}?

Problem D: Given $q \in U$, show that the linear map f_q defined above is a 3-dimensional rotation. A rotation on $\mathbb{R}^3$ is specified by a rotation axis and a rotation angle. Can you find the axis and the angle of the rotation for f_q? Does your answer for f_q^{-1} make sense when view them as rotations?

RULES:

- The competition is open to all *undergraduate* UNO students and it is supervised by *Upper Curriculum Committee* of the Mathematics Department.
- Submit your solutions to Andrzej Rosłanowski in DSC 222 or to his mailbox.
- Every nontrivial step/claim in your solution must justified. You may cite/quote a result from your textbook, past problems of the week and other widely available sources. In each case you have to give full reference.
- There are no partial credits, so rather err on the side of caution and provide more explanations than less. If you are not sure that your sources/references are appropriate, please include the complete relevant proofs from there.
- Your answers should be be written clearly and legibly. We reserve the right to refuse grading your work if it is difficult to read it.
- The winners of Fall 2018 edition of POW will be determined at the end of the semester based on the number of correct solutions submitted.
- Problems will be posted by Friday 5pm and the solutions are due by the following Friday 12 noon.

PRIZES:

- Winners will receive books published by the American Mathematical Society. The titles actually awarded will be selected in cooperation with the awardees.
- Everybody scoring in the POW Competition qualifies for the grand finale:
  $\frac{\pi}{2}$ *Mathematical Competition.*