Solution to Problem ♡–1

Are there rational numbers x, y, z, t such that

$$1+\sqrt{2}=(x+y\sqrt{2})^2+(z+t\sqrt{2})^2 \quad ?$$

Answer: No, there are no such rationals x, y, z, t. To show this, suppose towards contradiction that $x, y, z, t \in \mathbb{Q}$ satisfy

$$1+\sqrt{2}=(x+y\sqrt{2})^2+(z+t\sqrt{2})^2.$$

Then also

$$1-(x^2+z^2+2y^2+2t^2)=\sqrt{2}\cdot(2xy+2zt-1).$$

and hence, as $\sqrt{2}$ is not rational, we conclude that both sides of the above equality are 0. Thus

$$1=(x^2+z^2+2y^2+2t^2) \quad \text{and} \quad 2xy+2zt=1.$$

Consequently,

$$(x-y\sqrt{2})^2+(z-t\sqrt{2})^2=1-\sqrt{2}<0,$$

an obvious contradiction.

Correct solutions were received from :

(1) Sarah McCarty POW 1: ♡ Total Score so far: ♡