Solution to Problems ♡–5

Problem A: *For $x > 0$, show that*

$$\arctan(x) > \frac{3x}{1+2\sqrt{1+x^2}}.$$

Answer: Consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by formula

$$f(x) = \arctan(x) - \frac{3x}{1+2\sqrt{1+x^2}}.$$

The function f is differentiable on $\mathbb{R}$ and

$$f'(x) = \frac{1}{1+x^2} - \frac{3(1+2\sqrt{1+x^2}) - 3x \cdot \frac{2x}{\sqrt{1+x^2}}}{(1+2\sqrt{1+x^2})^2} =$$

$$\frac{(1+2\sqrt{1+x^2})^2 - 3(1+x^2) - 6(1+x^2)^{3/2} + 6x^2\sqrt{1+x^2}}{(1+x^2)\cdot(1+2\sqrt{1+x^2})^2} =$$

$$\frac{1+4\sqrt{1+x^2}+4(1+x^2)-3(1+x^2)-6\sqrt{1+x^2}\big((1+x^2)-x^2\big)}{(1+x^2)\cdot(1+2\sqrt{1+x^2})^2} =$$

$$\frac{1+4\sqrt{1+x^2}+(1+x^2)-6\sqrt{1+x^2}}{(1+x^2)\cdot(1+2\sqrt{1+x^2})^2} = \frac{\big(1-\sqrt{1+x^2}\big)^2}{(1+x^2)\cdot(1+2\sqrt{1+x^2})^2}.$$

Now it should be clear that $f'(x) > 0$ for all $x \in \mathbb{R}$ and the function f is strictly increasing on $\mathbb{R}$. Since $f(0) = 0$, we may conclude that $f(x) > 0$ for all $x > 0$, i.e.,

$$\arctan(x) > \frac{3x}{1+2\sqrt{1+x^2}}$$

for all $x > 0$.

CORRECT SOLUTION WAS RECEIVED FROM :

(1) CODY ANDERSON — POW 5A: ♡
(2) BRAD TUTTLE — POW 5A: ♡

Problem B: *Show that for all positive real numbers x, y the following inequality holds:*

$$x^y + y^x > 1.$$

Answer: The inequality is obvious if $x \geq 1$ or $y \geq 1$. So assume that $x, y \in (0,1)$. In view of the symmetry, it is enough to consider the case where $y \leq x$. Let $t = y/x \in (0,1]$. We have

$$x^y + y^x = x^{tx} + (tx)^x = \big(x^x\big)^t + t^x x^x.$$

Since the function $g : (0,\infty) \longrightarrow (0,\infty) : x \mapsto x^x$ attains its minimum value $e^{-1/e} = a$ at $\frac{1}{e}$ and since $t^x \geq t$ (as $0 < t \leq 1$, $0 < x < 1$), we see that

$$x^y + y^x \geq a^t + ta.$$

Consider a function $G : [0,\infty) \longrightarrow (0,\infty) : s \mapsto a^s + sa$. Clearly, G is differentiable and

$$G'(s) = \ln(a)a^s + a = e^{-1/e}\big(-\frac{1}{e}\cdot e^{\frac{-s}{e}+\frac{1}{e}} + 1\big) = e^{\frac{-1}{e}-1}\big(-e^{\frac{1-s}{e}} + e\big) > 0$$

for all $s \geq 0$. Therefore G is strictly increasing on $[0,\infty)$. Since $G(0) = 1$, we conclude that $G(t) > 1$ and hence

$$x^y + y^x \geq a^t + ta > 1.$$

Correct solution was received from :

(1) Brad Tuttle — POW 5B: ♡