Solution to Problems ♡–4

Problem A: *Does there exist a subset A of the plane $\mathbb{R}^2$ such that the orthogonal projection of A on any straight line in the plane has exactly 2017 distinct points?*

Answer: No, there is no such set. To show this let us assume toward contradiction that a set $A \subseteq \mathbb{R}^2$ has the property that the orthogonal projection of A on any straight line in the plane has exactly 2017 distinct points.

Pick any system OXY of Cartesian coordinates on the plane. Since the projections of our set A on each of the axes OX, OY have 2017 points, we conclude that the set A has at most $2017 \cdot 2017$ points.

Now consider all straight lines which go through at least 2 points from the set A. There are at most $\binom{2017^2}{2}$ (so finitely many) such lines. Consequently, there is an orthogonal projection which does not "glue" any two points of the set A, so the set A must have exactly 2017 points. But considering an orthogonal projection along a straight line going through two distinct points of our set we notice that this projection "glues" the two points on the line and thus the image of the set A in this projection has at most 2016 points, a contradiction.

Correct solution was received from :

(1) Brad Tuttle — POW 4A: ♡

Problem B: *Does there exists a set $B \subseteq \mathbb{R}^2$ which intersects every straight line in exactly 2017 points ?*

Answer: Yes, there are sets like this. We will build one using transfinite induction. Let $\mathfrak{c}$ be the cardinality of the continuum (treated as an ordinal number). Fix a list $\langle \ell_\alpha : \alpha < \mathfrak{c}\rangle$ of all straight lines in the plane.

By induction on $\alpha < \mathfrak{c}$ we construct sets $S_\alpha \subseteq \mathbb{R}^2$ so that the following inductive demands are satisfied:

$(*)_\alpha$ the union $\bigcup\limits_{\beta\leq\alpha} S_\beta$ contains exactly 2017 points from the line ℓ_α,

$(**)_\alpha$ the union $\bigcup\limits_{\beta\leq\alpha} S_\beta$ contains no 2018 colinear points, and

$(***)_\alpha$ S_α has at most 2017 elements.

Suppose that we have defined S_β for $\beta < \alpha$ so that the demands $(*)_\beta$–$(***)_\beta$ are satisfied. Let $A = \bigcup\limits_{\beta<\alpha} S_\beta$ and note that

$(\circledast)_1$ the set A contains no 2018 colinear points and

$(\circledast)_2$ the cardinality of A is smaller than the continuum $\mathfrak{c}$, so

$(\circledast)_3$ the family $\mathcal{L}$ of all straight lines passing through at least two points of A is of size smaller than $\mathfrak{c}$.

It follows from $(\circledast)_1$ that the intersection $\ell_\alpha \cap A$ has at most 2017 points. If the intersection $\ell_\alpha \cap A$ has (exactly) 2017 points, then we set $S_\alpha = \emptyset$. If this intersection has exactly $2017 - k$ points, with $0 < k \leq 2017$, then we may use $(\circledast)_3$ to choose points $x_1, \ldots x_k \in \ell_\alpha$ such that $x_1, \ldots, x_k \notin \ell$ for any $\ell \in \mathcal{L}$. We set $S_\alpha = \{x_1, \ldots, x_k\}$. One easily verifies that in each case the demands $(*)_\alpha$–$(***)_\alpha$ are satisfied.

After the above construction is carried out we let $B = \bigcup\limits_{\alpha<\mathfrak{c}} S_\alpha$. It follows from $(**)_\alpha$ (for $\alpha < \mathfrak{c}$) that the set B contains no 2018 colinear points. By $(*)_\alpha$ (for $\alpha < \mathfrak{c}$) it intersects each line at exactly 2017 points.

Intrigued by this solution? Would like to learn more about set theoretic methods in mathematics? Talk to Andrzej Rosłanowski about possible course/seminar "Set Theory for a working mathematician".

Correct solution was received from :

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