Solution to Problems ♡–3

Problem A: *Let x, y, and z be roots of the equation $x^3 + ax^2 + bx - ab = 0$, where a and b are real numbers. Find $(x+y)(x+z)(y+z)$. (You have to show all your work and/or justify your answer.)*

Answer: By the Viete's formulas we have:

$$-a = x + y + z, \quad b = xy + xz + yz, \quad \text{and } ab = xyz.$$

From the identity

$$(x+y+z)(xy+xz+yz) - xyz \equiv (x+y)(x+z)(y+z)$$

it follows that $(x+y)(x+z)(y+z) = -2ab$.

Correct solution was received from :

(1) Brad Tuttle POW 3A: ♡

Problem B: *Prove that if* $\frac{1}{a+b+c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$*, then*

$$\frac{1}{a^n + b^n + c^n} = \frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} \quad \textit{for any odd } n.$$

Answer: It follows from the equation

$$\frac{1}{a+b+c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

that

$$abc = (a+b+c)(ab+ac+bc).$$

Hence

$$0 = (a+b+c)(ab+ac+bc) - abc = (a+b)(a+c)(b+c),$$

and $a = -b$ or $a = -c$ or $b = -c$.

If $a = -b$, then

$$\frac{1}{a+b+c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{c}$$

and

$$\frac{1}{a^n + b^n + c^n} = \frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{c^n}$$

for any odd n. Same in other cases.

Correct solution was received from :

(1) Melissa Riley POW 3B: ♡ Total Score so far: ♡

(2) Brad Tuttle POW 3B: ♡ Total Score so far: ♡♡♡♡