Solution to Problems ♡–2

Problem A: *Suppose that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is nonconstant, periodic and has at least one continuity point. Prove that f has a smallest positive period, the so called* fundamental period.

Answer: Let P be the set of all positive periods of the function f. Suppose towards contradiction that the function f does not have the smallest positive period, that is the set P has no smallest element.

Note that if $t_1 < t_2$ are from P, then also $t_2 - t_1 \in P$, as

$$f\big(x + (t_2 - t_1)\big) = f\big((x - t_1) + t_2\big) = f\big(x - t_1\big) = f(x).$$

Also, if $t \in P$ and $n > 0$ is a natural number, then $nt \in P$ and $-nt$ is a period of f as well.

Since the set P is bounded from below and non-empty, it has the infimum. We may choose a (strictly) decreasing sequence $(s_n)_{n=1}^{\infty}$ of elements of P converging to $\inf(P)$. Then $T_n \stackrel{\text{def}}{=} s_n - s_{n+1} \in P$, $T_n > 0$ and $\lim\limits_{n\to\infty} T_n = 0$.

Let $x_0 \in \mathbb{R}$ be a point of continuity of f. Since f is not constant, there is $x_1 \neq x_2$ such that $f(x_1) \neq f(x_0)$. Put $\varepsilon_0 = \frac{|f(x_0)-f(x_1)|}{2} > 0$ and apply the continuity of f at x_0 to find $\delta_0 > 0$ such that

$$(\forall x \in \mathbb{R})(|x - x_0| < \delta_0 \ \Rightarrow \ |f(x) - f(x_0)| < \varepsilon_0).$$

Choose a natural number n_0 such that $0 < T_{n_0} < \delta_0/2$ (every sufficiently large integer will do). Then at least one of the numbers kT_{n_0}, $k \in \mathbb{Z}$, belongs to the interval $(x_0 - x_1 - \delta_0, x_0 - x_1 + \delta_0)$. Consequently, $x_1 + kT_{n_0} \in (x_0 - \delta, x_0 + \delta)$ and (by the choice of δ_0)

$$|f(x_1) - f(x_0)| = |f(x_1 + kT_{n_0}) - f(x_0)| < \varepsilon_0 < |f(x_1) - f(x_0)|,$$

a contradiction.

Correct solution was received from :

(1) Brad Tuttle

Problem B: *Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and periodic with period $t > 0$. Prove that there is $x_0 \in \mathbb{R}$ such that*

$$f\big(x_0 + \frac{t}{2}\big) = f(x_0).$$

Answer: Define the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(x) = f\big(x + \frac{t}{2}\big) - f(x) \qquad \text{for } x \in \mathbb{R}.$$

Then g is continuous on $\mathbb{R}$ and

- $g(0) = f\big(\frac{t}{2}\big) - f(0)$, and
- $g\big(\frac{t}{2}\big) = f(t) - f\big(\frac{t}{2}\big) = f(0) - f\big(\frac{t}{2}\big)$.

If $g(0) = 0$ then $f\big(0 + \frac{t}{2}\big) = f(0)$, so $x_0 = 0$ is as required. Otherwise, $g(0)$ and $g\big(\frac{t}{2}\big)$ have opposite signs so by the Intermediate Value Theorem there is $x_0 \in \big(0, \frac{t}{2}\big)$ such that $g(x_0) = 0$. Then $f\big(x_0 + \frac{t}{2}\big) = f(x_0)$.

Correct solutions were received from :

1. Cody Anderson — POW 2B: ♡ — Total Score so far: ♡
2. Trevor Pintzien — POW 2B: ♡ — Total Score so far: ♡
3. Brad Tuttle — POW 2B: ♡ — Total Score so far: ♡♡