Solution to Problems ♡–9

Problem A: *An old woman goes to market and a horse steps on her basket and crushes the eggs. The rider offers to pay for the damages and asks her how many eggs she had brought. She does not remember the exact number, but when she had taken them out two at a time, there was one egg left. The same happened when she picked them out three, four, five, and six at a time, but when she took them seven at a time they came out even. What is the smallest number of eggs she could have had?*

Answer: We will use the following theorem

Chineese Remainder Theorem (CRT): *Suppose that $m_1, m_2, \ldots, m_r$ are pairwise relatively prime positive integers, and let $a_1, a_2, \ldots, a_r$ be integers. Then the system of congruences,* $x \equiv a_i \pmod{m_i}$ *for* $1 \le i \le r$, *has a unique solution modulo* $M = m_1 \cdot m_2 \cdot \ldots \cdot m_r$, *which is given by:*

$$x \equiv a_1 M_1 y_1 + a_2 M_2 y_2 + \ldots + a_r M_r y_r \pmod{M},$$

where $M_i = M/m_i$ *and* $y_i \equiv (M_i)^{-1} \pmod{m_i}$ *for* $1 \le i \le r$.

The number of broken eggs, x, must satisfy:

$$\begin{array}{llll} x \equiv 1 \mod 2, & x \equiv 1 \mod 3, & & x \equiv 1 \mod 4, \\ x \equiv 1 \mod 5, & x \equiv 1 \mod 6 & \text{and} & x \equiv 0 \mod 7. \end{array}$$

The first congruence says that x is odd, so we shall keep this in mind and ignore this congruence. To use Chineese Remainder Theorem, we will also omit the congruence $x \equiv 1 \mod 6$ so that the moduli of the remaining congruences (3, 4, 5 and 7) are relatively prime in pairs. CRT indicates that there is a unique solution modulo 420 ($= 3 \cdot 4 \cdot 5 \cdot 7$), which is calculated as follows. We let

$$\begin{array}{ll} M_3 = 420/3 = 140, & y_3 \equiv (140)^{-1} \mod 3 = 2, \\ M_4 = 420/4 = 105, & y_4 \equiv (105)^{-1} \mod 4 = 1, \\ M_5 = 420/5 = 84, & y_5 \equiv (84)^{-1} \mod 5 = 4, \\ M_7 = 420/7 = 60, & y_7 \equiv (60)^{-1} \mod 7 = 2 \end{array}$$

and

$$x \equiv 1(140)(2) + 1(105)(1) + 1(84)(4) + 0(60)(2) \mod 420 = 301.$$

As this value of x is odd and satisfies $x \equiv 1 \mod 6$, it is the smallest solution of the broken eggs problem.

Correct solution was received from :

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(2) David Cavanaugh — POW 10A: ♡
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Problem B: *For a positive integer m let $S(m)$ denote the some of all digits of m (in the decimal representation). Consider the sequence $(a_n)_{n=1}^{\infty}$ defined recursively as follows:*

$$a_1 = 2017^{2017}, \qquad a_{n+1} = S(a_n).$$

What is a_{2017} ?

Answer: We note that

$$a_1 = 2017^{2017} < \left(10^4\right)^{2017} = 10^{8068},$$

and hence $a_2 = S(a_1) \leq 9 \cdot 8068 = 72612 < 10^5$. Consequently, $a_3 = S(a_3) \leq 9 \cdot 5 = 45$ and $a_4 = S(a_4) \leq 3 + 9 = 12$. Finally, $1 \leq a_5 = S(a_4) \leq 9$.

We know also that $m \equiv S(m) \mod 9$ for any integer m, so

$$a_1 \equiv a_5 \mod 9.$$

Noticing that $2017 \equiv 1 \mod 9$, so $2017^{2017} \equiv 1 \mod 9$, we get

$$a_1 \equiv 1 \mod 9.$$

Consequently, $a_5 \equiv 1 \mod 9$ and thus $a_5 = 1$. Then clearly for each $k \geq 5$ we also have $a_k = 1$.

Correct solution was received from :

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